# THE SCHWARZIAN DERIVATIVE AND MEASURED LAMINATIONS ON RIEMANN SURFACES

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Abstract. A holomorphic quadratic differential on a hyperbolic Riemann surface has an associated measured foliation, which can be straightened to yield a measured geodesic lamination. On the other hand, a quadratic differential can be considered as the Schwarzian derivative of a $\mathbb{CP}^1$ structure, to which one can naturally associate another measured geodesic lamination using grafting.

We compare these two relationships between quadratic differentials and measured geodesic laminations, each of which yields a homeomorphism $\mathscr{ML}(S) \to Q(X)$ for each conformal structure $X$ on a compact surface $S$. We show that these maps are nearly the same, differing by a multiplicative factor of $-2$ and an error term of lower order than the maps themselves (which we bound explicitly).

As an application we show that the Schwarzian derivative of a $\mathbb{CP}^1$ structure with Fuchsian holonomy is close to a $2\pi$-integral Jenkins-Strebel differential. We also study compactifications of the space of $\mathbb{CP}^1$ structures using the Schwarzian derivative and grafting coordinates; we show that the natural map between these extends to the boundary of each fiber over Teichmüller space, and we describe this extension.

## 1. Introduction

In this paper we compare two natural homeomorphisms between the vector space $Q(X)$ of holomorphic quadratic differentials on a compact Riemann surface $X$ and the $PL$-manifold $\mathscr{ML}(S)$ of measured laminations on the differentiable surface $S$ underlying $X$.

The first of these two homeomorphisms is a product of 2-dimensional geometry–specifically, the theory of measured foliations on Riemann surfaces. A holomorphic quadratic differential on $X$ defines a measured foliation whose leaves are its horizontal trajectories [Str2]. Hubbard and Masur showed that each equivalence class of measured foliations is obtained uniquely in this way, so that the resulting map from $Q(X)$ to the space $\mathscr{MF}(S)$ of equivalence classes of measured foliations is a homeomorphism [HM].

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On the other hand, to a measured foliation one can associate a measured geodesic lamination having the same intersection properties with simple closed curves. Thurston showed that the resulting map $\mathscr{MF}(S) \to \mathscr{ML}(S)$ is a homeomorphism (see [Lev] for a detailed treatment). Combining this homeomorphism and the Hubbard-Masur map, we obtain a homeomorphism $\phi_F : \mathscr{ML}(S) \to Q(X)$, which we call the *foliation map*.

We will compare this map to another homeomorphism between $\mathscr{ML}(S)$ and $Q(X)$ arising in the theory of complex projective ($\mathbb{CP}^1$) structures on Riemann surfaces. Let $P(X)$ denote the space of marked $\mathbb{CP}^1$ surfaces with underlying Riemann surface $X$. It is a classical result that taking the Schwarzian derivatives of the chart maps induces a homeomorphism $P(X) \to Q(X)$, providing a complex-analytic parameterization of this moduli space.

Using Thurston's theory of $\mathbb{CP}^1$ structures and a result of Scannell and Wolf on grafting, one can obtain a more hyperbolic-geometric description of $P(X)$, leading to a homeomorphism $P(X) \to \mathscr{ML}(S)$. Here the lamination $\lambda \in \mathscr{ML}(S)$ associated to a projective structure $Z \in P(X)$ records the bending of a locally convex pleated plane in $\mathbb{H}^3$ that is the "convex hull" of the development of $\widetilde{Z}$ to $\mathbb{CP}^1$ [KT].

As before we combine the two homeomorphisms to obtain a homeomorphism $\phi_T : \mathscr{ML}(S) \to Q(X)$, which we call the *Thurston map*.

Our goal is to show that in spite of the lack of any apparent geometric relationship between measured foliations and complex projective structures, the maps $\phi_T$ and $\phi_F$ are approximately multiples of one another, up to an error term of smaller order than either map. More precisely, we have:

**Theorem 1.1.** *Fix a conformal structure $X$ on a compact surface $S$. Then for all $\lambda \in \mathscr{ML}(S)$, the foliation map $\phi_F : \mathscr{ML}(S) \to Q(X)$ and the Thurston map $\phi_T : \mathscr{ML}(S) \to Q(X)$ satisfy*

$$\|2\phi_T(\lambda) + \phi_F(\lambda)\|_{L^1(X)} \leq C(X) \left(1 + \sqrt{\|\phi_F(\lambda)\|_{L^1(X)}}\right)$$

*where $C(X)$ is a constant that depends only on $X$.*

**The Schwarzian derivative.** The constructions used to define the foliation map $\phi_F$ and the Thurston parameterization of $\mathbb{CP}^1$ structures are essentially geometric, and can be understood as part of the rich interplay between the theory of Riemann surfaces and hyperbolic geometry in two and three dimensions. The Schwarzian derivative seems more analytic than geometric, however, and relating the quadratic differential obtained from the charts of a $\mathbb{CP}^1$ structure (using the Schwarzian) to geometric data accounts for a significant part of the proof of Theorem 1.1.

Relationships between the Schwarzian derivative and geometry have been explored by a number of authors, using curvature of curves in the plane [Fla] [CDO2], curvature of surfaces in hyperbolic space [Eps], Lorentzian geometry [DO] [OT], and osculating Möbius transformations [Thu] [And].

Furthermore, there are a number of different ways to generalize the classical Schwarzian derivative to other kinds of maps, more general domain and range spaces, or both (e.g. [Ahl] [GF] [BO]).

The proof of Theorem 1.1 turns on the decomposition of the Schwarzian derivative of a $\mathbb{CP}^1$ structure into a sum of two parts (Theorem 7.1), one of which is manifestly geometric, and another which is "small", having norm bounded by a constant depending only on $X$. This decomposition is a product of the Osgood-Stowe theory of the Schwarzian *tensor*–a particular generalization of the Schwarzian derivative that measures the difference between two conformally equivalent Riemannian metrics [OS]. Much as the Schwarzian derivative of a composition of holomorphic maps can be decomposed using the cocycle property (or "chain rule"), the Schwarzian tensor allows us to decompose the quadratic differential associated to a $\mathbb{CP}^1$ structure by finding a conformal metric that interpolates those of the domain and range of the projective charts. In our case, the appropriate interpolating metric is the *Thurston metric* associated to a grafted surface, which is a sort of Kobayashi metric in the category of $\mathbb{CP}^1$ surfaces (see §5).

We apply this decomposition of the Schwarzian to $\phi_T(\lambda)$, the Schwarzian derivative of the $\mathbb{CP}^1$ structure on $X$ with grafting lamination $\lambda \in \mathscr{ML}(S)$. The two terms from this decomposition of are then analyzed separately, ultimately leading to a proof of Theorem 1.1.

**Harmonic maps.** The first part of the decomposition of $\phi_T(\lambda)$ is the *grafting differential* $\Phi(\lambda)$, a non-holomorphic quadratic differential on $X$ whose trajectories foliate the grafted part of the $\mathbb{CP}^1$ structure. The grafting differential can also be defined as (a multiple of) the Hopf differential of the *collapsing map* of the $\mathbb{CP}^1$ structure, which is a generalization of the retraction of a set in $\mathbb{CP}^1$ to the boundary of the hyperbolic convex hull of its complement.

Using harmonic maps techniques, one can bound the difference between the grafting differential and Hubbard-Masur differential $\phi_F(\lambda)$ in terms of the extremal length of $\lambda$ (see §5, also [D2] [D1]):

$$\|\Phi(\lambda) - \phi_F(\lambda)\|_{L^1(X)} \leq C\,(1 + E(\lambda, X)^{\frac{1}{2}})$$

Here the constant $C$ depends only on the topological type of $X$ (see §5). Geometrically, this makes sense: the grafting differential is holomorphic where it is nonzero, which is most of $X$ when $\lambda$ is large (see Figure 2 in §10); thus it should be close to the unique holomorphic differential $\phi_F(\lambda)$ with the same trajectory structure.

**Analytic methods.** The second part of the decomposition of $\phi_T(\lambda)$ is the Schwarzian tensor of the Thurston metric of the $\mathbb{CP}^1$ structure relative to the hyperbolic metric of $X$. We complete the proof of Theorem 1.1 by showing that this non-holomorphic quadratic differential is bounded, using analysis and the properties of the Thurston metric.

Essential to this bound is the fact that the Schwarzian tensor of a conformal metric (relative to any fixed background metric) depends only on the 2-jet of its density function, and thus the norm of this tensor is controlled by the norm of the density function in an appropriate Sobolev space. Using standard elliptic theory, it is actually enough to bound the Laplacian of the density function, which is essentially the curvature 2-form of the Thurston metric.

On the other hand, the curvature of the Thurston metric can be understood using the measured lamination $\lambda$ and the properties of grafting. We ultimately show that the Thurston metric's curvature is concentrated near a finite set of points, except for an exponentially small portion that may diffuse into the rest of $X$. This is enough to bound the Schwarzian tensor on a hyperbolic disk of definite size, and then, using a compactness argument, on all of $X$.

**Applications.** As an application of Theorem 1.1, we show that the countably many $\mathbb{CP}^1$ structures on $X \in \mathscr{T}(S)$ with Fuchsian holonomy are close to the $2\pi$-integral Jenkins-Strebel differentials in $Q(X)$ (Theorem 13.1). In particular these *Fuchsian centers* are arranged in a regular pattern, up to an error term of smaller order than their norms (though the difference may be unbounded for a sequence of centers going to infinity). Numerical experiments illustrating this effect are presented in §13.

We also apply the main theorem to the fiberwise compactification of the space of $\mathbb{CP}^1$ structures induced by the Schwarzian map $P(X) \xrightarrow{\sim} Q(X)$, and show that the map to the grafting coordinates $P(X) \hookrightarrow \mathscr{ML}(S) \times \mathscr{T}(S)$ has a natural continuous extension (Theorem 14.2). This extends the results of [D1], where this boundary of $P(X)$ was studied independently of the Schwarzian parameterization using the *antipodal involution* $i_X : \mathbb{P}\mathscr{ML}(S) \to \mathbb{P}\mathscr{ML}(S)$.

**Outline of the paper.**

**§2** presents some background material on conformal metrics and tensors on Riemann surfaces, which are the main objects of study in the proof of Theorem 1.1.

**§3** describes the Hubbard-Masur construction of a homeomorphism between $Q(X)$ and $\mathscr{MF}(S)$, and the associated *foliation map* $\phi_F : \mathscr{ML}(S) \to Q(X)$.

**§4** describes another connection between $\mathscr{ML}(S)$ and $Q(X)$ using Thurston's theory of grafting and $\mathbb{CP}^1$ structures on Riemann surfaces and results of Scannell-Wolf on grafting. The result is the *Thurston map* $\phi_T : \mathscr{ML}(S) \to Q(X)$.

**§5** continues our study of grafting and projective structures by introducing the *Thurston metric* and the *grafting differential*. These natural geometric objects are analogous to the singular Euclidean metric and the holomorphic quadratic differential that arise in the Hubbard-Masur construction.

**§6** introduces the Osgood-Stowe Schwarzian tensor, a generalization of the Schwarzian derivative that we use to relate the Schwarzian of a projective structure to the analytic properties of conformal metrics.

**§7** establishes a decomposition of the Schwarzian of a projective structure (Theorem 7.1) into the sum of the grafting differential and the Schwarzian tensor of the Thurston metric. This decomposition is the main conceptual step toward the proof of the main theorem.

**§§8-9** begin our analytic study of the Thurston metric by deriving certain regularity properties of nonpositively curved conformal metrics on Riemann surfaces (of which the Thurston metric is an example).

**§§10-11** contain the key estimates on the Thurston metric, showing that its curvature concentrates near a finite set of points (Theorem 10.1) away from which its Schwarzian tensor is bounded (Theorem 11.4).

**§12** contains the proof Theorem 1.1, which combines the bound on the Schwarzian tensor of the Thurston metric, the decomposition of the Schwarzian of a projective structure, and properties of the grafting differential (Theorems 11.4, 7.1, and 5.2, respectively).

**§§13-14** present the applications of Theorem 1.1 mentioned above—locating Fuchsian centers in $P(X)$ (and related numerical experiments), and the continuous extension of the grafting coordinates $P(X) \hookrightarrow \mathscr{ML}(S) \times \mathscr{T}(S)$ to respective compactifications.

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**Notational conventions.** In what follows, $X$ denotes a compact (except in §13) hyperbolic Riemann surface and $S$ the underlying differentiable surface of genus $g$ and Euler characteristic $\chi = 2 - 2g$.

The expression $C(a, b, \ldots)$ is used to indicate that an unspecified constant $C$ depends on quantities $a, b, \ldots$, one of which is typically a conformal structure $X$.

## 2. Conformal metrics and tensors

We briefly recall some constructions related to conformal metrics and tensors on Riemann surfaces that will be used in the sequel. On a fixed compact Riemann surface $X$, choose a complex line bundle $\Sigma$ with $\Sigma^2 = K_X$. This allows us to define the bundle $S_{i,j} = S_{i,j}(X)$ of differentials of type $(i, j)$ for all $(i, j) \in (\frac{1}{2}\mathbb{Z})^2$:

$$S_{i,j} = \Sigma^{2i}\bar{\Sigma}^{2j}$$

We will be most interested in $S_{\frac{1}{2},\frac{1}{2}}$, whose sections include *conformal metrics* on $X$, which in a coordinate chart have the form

$$\rho(z)\,|dz|$$

where $\rho$ is a nonnegative function. Such a conformal metric determines length and area functions,

$$\ell(\gamma,\rho) = \int_\gamma \rho \quad \text{where} \quad \gamma : [0,1] \to X$$
$$A(\Omega,\rho) = \int_\Omega \rho^2 \quad \text{where} \quad \Omega \subset X$$

which, modulo sufficient regularity and positivity of $\rho$, make $X$ into a geodesic metric space. The properties of a particular class of these metrics will be studied further in §§8-9.

The $L^p$ norm of a section of $S_{i,j}$ is well-defined independent of any background metric on $X$ when $p(i+j)=2$. For example, the $L^2$ norm on $S_{\frac{1}{2},\frac{1}{2}}$ corresponds to the area of a conformal metric. Given a conformal metric $\rho \in L^2(S_{\frac{1}{2},\frac{1}{2}})$, we can also define the $L^p$ norm of a section $\xi$ of $S_{i,j}$ with respect to $\rho$:

$$\|\xi\|_{L^p(S_{i,j},\rho)} = \int_X |\xi|^p \rho^{2-p(i+j)}.$$

When the tensor type is understood and a background metric $\rho$ is fixed (or unnecessary) we will abbreviate this norm as $\|\xi\|_{L^p(X)}$ or simply $\|\xi\|_p$.

A pullback construction will provide most of our examples of conformal metrics; specifically, given a smooth map $f : X \to (M,g)$, where $M$ is a Riemannian manifold, the pullback metric $f^*(g)$ need not lie in the conformal class of $X$, however it can be decomposed relative to this conformal structure as follows:

$$f^*(g) = \Phi(f) + \rho(f)^2 + \overline{\Phi(f)}$$

Here $\Phi(f) \in \Gamma(S_{2,0})$ is the *Hopf differential* of $f$ and $\rho(f) \in \Gamma(S_{\frac{1}{2},\frac{1}{2}})$ is a conformal metric that is the "isotropic part" of the pullback of the line element of $g$. Of course the smoothness assumption on $f$ may be relaxed considerably; the natural regularity class for our purposes is the space $W^{1,2}(X,M)$ of Sobolev maps into $M$ with $L^2$ derivatives. For such maps, the resulting conformal metric $\rho$ is in $L^2(S_{\frac{1}{2},\frac{1}{2}})$ and the Hopf differential $\Phi$ lies in $L^1(S_{2,0})$.

## 3. The Hubbard-Masur construction

In this section we review the first of two relationships between measured laminations and quadratic differentials that we will explore. A result of Hubbard and Masur is essential to this relationship.

Let $Q(X) \subset L^1(S_{2,0}(X))$ denote the space of holomorphic quadratic differentials on $X$. Each differential $\phi \in Q(X)$ has an associated (singular) measured foliation $\mathscr{F}(\phi)$ whose leaves integrate the distribution of vectors

$v \in TX$ satisfying $\phi(v) \geq 0$. There is also a well-known homeomorphism $\mathscr{MF}(S) \simeq \mathscr{ML}(S)$ between the spaces of (measure equivalence classes of) measured foliations and measured geodesic laminations on a hyperbolic surface. Roughly speaking, to obtain a lamination from a measured foliation, one replaces the nonsingular leaves of the foliation with geodesic representatives for the hyperbolic metric on $X$ (for a detailed account, see [Lev]). Thus we obtain a map $\Lambda : Q(X) \to \mathscr{ML}(S)$.

Hubbard and Masur showed that $\mathscr{F}$ (and thus also $\Lambda$) is a homeomorphism [HM]; in other words, given any measure equivalence class of measured foliations $\mathscr{F}_0$, there is a unique holomorphic quadratic differential $\phi$ with $\mathscr{F}(\phi) \sim \mathscr{F}_0$. (For other perspectives on this result see [Ker] [Gar2] [Wol5], and for the special case of foliations with closed trajectories see e.g. [Jen] [Str1] [Gar1] [MS] [Wol4].) We will be interested in the inverse homeomorphism

$$\phi_F : \mathscr{ML}(S) \to Q(X)$$

which we call the *foliation map*, as it associates to $\lambda \in \mathscr{ML}(S)$ a quadratic differential whose foliation has prescribed measure properties.

The foliation map $\phi_F : \mathscr{ML}(S) \to Q(X)$ is well-behaved with respect to natural structures on the spaces $\mathscr{ML}(S)$ and $Q(X)$; it preserves basepoints (i.e. $\phi_F(0) = 0$, where $0 \in \mathscr{ML}(S)$ is the empty lamination), and is homogeneous of degree 2 on rays in $\mathscr{ML}(S)$.

## 4. Grafting and $\mathbb{CP}^1$ structures

One might say that the foliation homeomorphism $\phi_F : \mathscr{ML}(S) \to Q(X)$ exists because both $\mathscr{ML}(S)$ and $Q(X)$ are models for the space $\mathscr{MF}(S)$ of measured foliation classes on $S$; one model comes from hyperbolic geometry ($\mathscr{ML}(S) \simeq \mathscr{MF}(S)$), while the other comes from the singular Euclidean geometry of a quadratic differential ($Q(X) \simeq \mathscr{MF}(S)$).

In a similar vein, we now construct a homeomorphism $\phi_T : \mathscr{ML}(S) \to Q(X)$, which we call the *Thurston map*, by showing that both $\mathscr{ML}(S)$ and $Q(X)$ are naturally homeomorphic to the space $P(X)$ of complex projective structures on $X$. The identification of $\mathscr{ML}(S)$ with $P(X)$ will involve hyperbolic geometry (in $\mathbb{H}^3$) and Thurston's projective version of *grafting*, while that of $Q(X)$ with $P(X)$ uses the Schwarzian derivative. We begin with a few generalities on $\mathbb{CP}^1$-structures.

A complex projective structure on $S$ is an atlas of charts with values in $\mathbb{CP}^1$ and Möbius transition functions. The space $\mathscr{P}(S)$ of marked $\mathbb{CP}^1$ structures fibers over Teichmüller space by the map $\pi : \mathscr{P}(S) \to \mathscr{T}(S)$ which records the underlying complex structure. When the underlying complex structure of a $\mathbb{CP}^1$ surface is $X$, we say it is a $\mathbb{CP}^1$ structure *on* $X$.

Let $Z$ be a projective structure on $X$. The chart maps of $Z$ can be analytically continued on the universal cover $\widetilde{X} \simeq \mathbb{D}$ to give a locally univalent holomorphic map $f : \widetilde{X} \to \mathbb{CP}^1$, called the *developing map*. This map is not unique, but any two such maps differ by composition with a Möbius transformation. When restricted to any open set on which it is univalent, the developing map is a projective chart.

Because the projective structure on $\widetilde{X}$ induced by lifting $Z$ is invariant under the action of $\pi_1(S)$ by deck transformations, for any $\gamma \in \pi_1(S)$ and $z \in \widetilde{X}$, the germs $f_z$ and $f_{\gamma z}$ differ by composition with a Möbius transformation $A_\gamma$. The map $\gamma \mapsto A_\gamma$ defines a homomorphism $\eta(Z) : \pi_1(S) \to \mathrm{PSL}_2(\mathbb{C})$, called the *holonomy representation* of $Z$, which is unique up to conjugation.

To obtain a concrete realization of the fiber $P(X) = \pi^{-1}(X)$, we use the *Schwarzian derivative*, a Möbius-invariant differential operator on locally injective holomorphic maps to $\mathbb{CP}^1$:

$$S(f) = \left[ \left( \frac{f''(z)}{f'(z)} \right)' - \frac{1}{2} \left( \frac{f''(z)}{f'(z)} \right)^2 \right] dz^2$$

By its Möbius invariance, the Schwarzian derivatives of the charts of a $\mathbb{CP}^1$ structure on $X$ (viewed as maps from subsets of $\mathbb{D} \simeq \mathbb{H}^2 \simeq \widetilde{X}$ to $\mathbb{CP}^1$) join together to form a holomorphic quadratic differential on $X$. Equivalently, the Schwarzian derivative of the developing map $S(f) \in Q(\mathbb{D})$ is invariant under the action of $\pi_1(S)$ by deck transformations of the universal covering $\mathbb{D} \to X$, and so it descends to a quadratic differential on $X$. The resulting map

$$P(X) \xrightarrow{S} Q(X)$$

is a homeomorphism, and in fact a biholomorphism with respect to the natural complex structure of $P(X)$. This is the *Poincaré parameterization* of $P(X)$ (see [Hej1], [Gun1]).

An alternate and more geometric approach to $\mathbb{CP}^1$ structures was described by Thurston using *grafting*, a cut-and-paste operation on hyperbolic Riemann surfaces (see e.g. [Mas] [Hej2] [KT] [SW]). The conformal grafting map

$$\mathrm{gr} : \mathscr{ML}(S) \times \mathscr{T}(S) \to \mathscr{T}(S)$$

sends the pair $(\lambda, Y)$ to a surface obtained by removing the geodesic lamination supporting $\lambda$ from $Y$ and replacing it with a "thickened lamination" that has a Euclidean structure realizing the measure of $\lambda$. The details of the construction are more easily explained when $\lambda$ is supported on a simple closed geodesic $\gamma$ with weight $t$, in which case $\gamma$ is simply replaced with the Euclidean cylinder $\gamma \times [0, t]$ to obtain the grafted surface $\mathrm{gr}_{t\gamma} Y$ (see Figure 1). As such weighted curves are dense in $\mathscr{ML}(S)$, the existence of the grafting operation in general can be reduced to a continuity property; for details, see [KT].

Thurston introduced a projective grafting map $\mathrm{Gr} : \mathscr{ML}(S) \times \mathscr{T}(S) \to \mathscr{P}(S)$ that puts a canonical projective structure on a grafted surface. This map associates to $(\lambda, Y)$ a projective structure on $\mathrm{gr}_\lambda Y$ whose "convex hull boundary" in $\mathbb{H}^3$ is the locally convex pleated surface obtained by bending $\widetilde{Y} \simeq \mathbb{H}^2$ along the lift of the measured lamination $\lambda$. Roughly speaking,

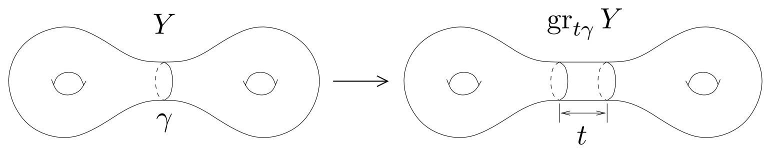


**Figure 1:** *The basic example of grafting.*

the Gauss map from this surface (which follows normal geodesic rays out to $\mathbb{CP}^1 = \partial_\infty \mathbb{H}^3$) provides a system of charts for this projective structure. This projective version of grafting is especially interesting because every projective structure can be obtained from grafting in exactly one way; that is,

$$\mathscr{ML}(S) \times \mathscr{T}(S) \xrightarrow{\mathrm{Gr}} \mathscr{P}(S)$$

is a homeomorphism. This is *Thurston's theorem*, a detailed proof of which can be found in [KT].

Thus grafting gives a geometric description of $\mathscr{P}(S)$, and we can associate to any projective surface a pair $(\lambda, Y) \in \mathscr{ML}(S) \times \mathscr{T}(S)$. However, it is not immediately clear how the fibers $P(X)$ with a fixed underlying conformal structure fit into this grafting picture. We will now describe how one of the grafting coordinates, the measured lamination $\lambda \in \mathscr{ML}(S)$, suffices to parameterize any such fiber.

Scannell and Wolf showed that for each $\lambda \in \mathscr{ML}(S)$ the conformal grafting map $\mathrm{gr}_\lambda : \mathscr{T}(S) \to \mathscr{T}(S)$ is a homeomorphism [SW]; we call the inverse homeomorphism *pruning* by $\lambda$:

$$\mathrm{pr}_\lambda = \mathrm{gr}_\lambda^{-1} : \mathscr{T}(S) \to \mathscr{T}(S). \tag{4.1}$$

Thus for each lamination $\lambda \in \mathscr{ML}(S)$, the hyperbolic surface

$$Y_\lambda = \mathrm{pr}_\lambda X$$

has the property that when grafted by $\lambda$, the resulting surface is conformally equivalent to $X$. If we use Thurston's projective extension of grafting, then the result of grafting $Y_\lambda$ by $\lambda$ is a projective structure on $X$, which we call $X(\lambda)$, i.e.

$$X(\lambda) = \mathrm{Gr}_\lambda \, \mathrm{pr}_\lambda X \in P(X).$$

The resulting map

$$\begin{aligned} \beta : \mathscr{ML}(S) &\to P(X) \\ \lambda &\mapsto X(\lambda) \end{aligned}$$

is evidently a homeomorphism, because its inverse is the composition of $\mathrm{Gr}^{-1}|_{P(X)} : P(X) \to \mathscr{ML}(S) \times \mathscr{T}(S)$ with the projection of $\mathscr{ML}(S) \times \mathscr{T}(S)$ onto the first factor.

By taking the Schwarzian of the projective structure $X(\lambda)$, we obtain the *Thurston map* $\phi_T : \mathscr{ML}(S) \xrightarrow{\sim} Q(X)$:

$$\begin{tikzcd} \mathscr{ML}(S) \arrow[r, "\beta"] \arrow[rr, bend right, "\phi_T"'] & P(X) \arrow[r, "S"] & Q(X) \end{tikzcd}$$

When compared to the foliation map, the Thurston map $\phi_T$ is somewhat opaque; it is a homeomorphism, and it is easily seen to map the empty lamination to the zero differential (i.e. $\phi_T(0) = 0$), but there is no reason expect this map to be homogeneous, or even to preserve rays (see §13 for numerical experiments suggesting that it does not). The main obstruction to an intuitive understanding of this map would seem to be the lack of a connection between the analytic definition of the Schwarzian derivative and geometric properties of the projective surface. To put it another way, the Schwarzian $\phi$ of a projective structure on $X$ has an associated foliation $\mathscr{F}(\phi)$, but there is no obvious relationship between the geometry of this foliation and that of the $\mathbb{CP}^1$ structure.

It is just such a relationship we hope to reveal by relating the Hubbard-Masur construction (and $\phi_F$) to the Thurston map $\phi_T$, if only approximately.

## 5. The Thurston metric and grafting differential

The foliation map $\phi_F : \mathscr{ML}(S) \to Q(X)$ associates a holomorphic quadratic differential $\phi_F(\lambda)$ to each measured lamination $\lambda$, which in turn gives a singular Euclidean conformal metric $|\phi_F(\lambda)|^{1/2}$ on $X$. This metric is the natural one in which to examine the measured foliation $\mathscr{F}(\phi_F(\lambda))$, and is extremal for this foliation class in the sense of extremal length.

We now start to develop a similar picture for the Thurston map. In this case each measured lamination $\lambda$ gives rise to a projective structure $X(\lambda) \in P(X)$ of the form $\mathrm{Gr}_\lambda Y_\lambda$ whose Schwarzian is $\phi_T(\lambda)$. The grafting construction presents $X(\lambda)$ as a union of two parts: The hyperbolic part, $X_{-1}(\lambda)$, which comes from $Y_\lambda$, and a Euclidean part, $X_0(\lambda)$, which is grafted into $Y_\lambda$ along the geodesic lamination $\lambda$.

There is a natural conformal metric $\rho_\lambda \in L^2(S_{\frac{1}{2},\frac{1}{2}}(X))$ on $X$ associated to the projective structure $X(\lambda)$, called the *Thurston metric*, which combines the hyperbolic metric on $X_{-1}(\lambda)$ and the Euclidean metric on $X_0(\lambda)$ (see [SW, §2.2], [KP]). For example, $\rho_0$ is the hyperbolic metric on $X$.

Another convenient description of the Thurston metric is as the "Kobayashi metric" in the category of $\mathbb{CP}^1$ surfaces and locally Möbius maps [Tan1, §2.1]. To make this precise, we define a projectively immersed disk in a $\mathbb{CP}^1$ surface $Z$ to be a locally Möbius map $\delta$ from the unit disk in $\mathbb{C}$ (with its canonical $\mathbb{CP}^1$ structure) to $Z$. Then the Thurston length of a tangent vector $v \in T_x Z$ is the minimum length it is assigned by the hyperbolic metric of $\mathbb{D}$ when pulled back via a projective immersion $\delta : \mathbb{D} \to Z$ with $\delta(0) = x$.

Since the hyperbolic metric $\rho_0$ is the ordinary Kobayashi metric for $X$, where length is obtained as an infimum over the class of holomorphic immersions of the disk (which is larger than the class of projective immersions), we conclude immediately from this definition that for all $\lambda \in \mathscr{ML}(S)$,

$$\rho_\lambda \geq \rho_0. \tag{5.1}$$

The Thurston metric is also related to the *collapsing map* $\kappa : X \to Y_\lambda$, which collapses the grafted part of a surface orthogonally onto the geodesic representative of $\lambda$ on $Y_\lambda$. This map is distance non-increasing for the Thurston metric on $X$, and the Thurston metric is pointwise the smallest conformal metric on $X$ with this property (since at every point there is a direction in which $\kappa$ is an isometry with respect to the Thurston metric [KP, Thm. 8.6], any smaller conformal metric on $X$ would make the map expand somewhere).

The Hopf differential $\Phi(\kappa) \in L^2(S_{2,0}(X))$ of the collapsing map has an associated partial measured foliation $\mathscr{F}(\Phi(\kappa))$ that is supported in the grafting locus $X_0(\lambda)$, and whose leaves are Euclidean geodesics. Let us define the *grafting differential* $\Phi(\lambda)$,

$$\Phi(\lambda) = 4\Phi(\kappa). \tag{5.2}$$

The normalization is chosen so that the partial measured foliation $\mathscr{F}(\Phi(\lambda))$ represents the measure equivalence class $\lambda$ (see [D1, §6]), i.e.

$$\mathscr{F}(\Phi(\lambda)) \sim \lambda,$$

where $a \sim b$ means that $a$ and $b$ have the same intersection numbers with all simple closed curves. Thus $\Phi(\lambda)$ and $\mathscr{F}(\Phi(\lambda))$ are to the Thurston metric much as a holomorphic differential $\phi$ and foliation $\mathscr{F}(\phi)$ are to the singular Euclidean metric $|\phi|^{1/2}$.

Much about the large-scale behavior of grafting and related objects can be determined using the fact that the collapsing map $\kappa : X \to Y_\lambda$ is nearly harmonic, i.e. it nearly minimizes energy in its homotopy class.

**Theorem 5.1** (Tanigawa, [Tan1, Thm. 3.4])**.** *Let $X = \mathrm{gr}_\lambda Y$ with collapsing map $\kappa : X \to Y$, and let $h : X \to Y$ denote the harmonic map compatible with the markings of $X$ and $Y$, and $\mathscr{E}(h)$ its energy. Then*

$$\mathscr{E}(h) \approx \mathscr{E}(\kappa) \approx \frac{1}{2}\ell(\lambda, Y) \approx \frac{1}{2}E(\lambda, X) = \frac{1}{2}\|\phi_F(\lambda)\|_1$$

*where $E(\lambda, X)$ is the extremal length of $\lambda$ on $X$ and $\ell(\lambda, Y)$ is the length of $\lambda$ with respect to the hyperbolic metric on $Y$. Here $A \approx B$ means that the difference $A - B$ is bounded by a constant that depends only on the topology of $X$.*

*Remark.* In [Tan1], Tanigawa establishes a set of inequalities relating $\mathscr{E}(\kappa) = \frac{1}{2}\ell(\lambda, Y) + 2\pi|\chi(S)|$, $\mathscr{E}(h)$, and $E(\lambda, X)$, from which the approximate equalities in Theorem 5.1 follow by algebra. See [D1, §7] for details.

Harmonic maps techniques can also be used to relate the grafting differential to the foliation map; in fact, we have:

**Theorem 5.2** ([D2, Thm. 10.1], [D1])**.** *For any $X \in \mathscr{T}(S)$ and $\lambda \in \mathscr{ML}(S)$, the holomorphic quadratic differential $\phi_F(\lambda) \in Q(X)$ and the grafting differential $\Phi(\lambda) \in L^1(S_{2,0}(X))$ satisfy*

$$\|\Phi(\lambda) - \phi_F(\lambda)\|_1 \leq C\left(1 + E(\lambda, X)^{\frac{1}{2}}\right).$$

Since Theorem 5.2 has an important role in the proof of Theorem 1.1, we take a moment to sketch the ideas behind it: First, a construction dual to that of the collapsing map gives a *co-collapsing map* $\hat{\kappa} : \widetilde{X} \to T_\lambda$ with Hopf differential $\Phi(\hat{\kappa}) = -\frac{1}{4}\Phi(\lambda)$, where $T_\lambda$ is the $\mathbb{R}$-tree dual to $\lambda$. An energy estimate in the spirit of Theorem 5.1 shows that the co-collapsing map is nearly harmonic, and by a theorem of Wolf, the Hopf differential of the harmonic map to $T_\lambda$ is $-\frac{1}{4}\phi_F(\lambda)$. Finally, an estimate of Korevaar-Schoen from [KS, §2.6] shows that a nearly-harmonic map to a tree (or indeed, any $CAT(0)$ metric space) has Hopf differential which is close to that of the harmonic map, leading to the specific bound in Theorem 5.2.

What is missing from this harmonic maps picture is any geometric control on the Schwarzian $\phi_T(\lambda)$. In the next section we discuss the Osgood-Stowe generalization of the Schwarzian derivative, which we then use in §7 to relate the Schwarzian and the grafting differential.

## 6. The Schwarzian derivative and Schwarzian tensor

In a precise sense, the Schwarzian derivative measures the extent to which a locally injective holomorphic map $f$ fails to be a Möbius transformation [Thu]; for example, $f$ is (the restriction of) a Möbius transformation if and only if $S(f) = 0$. The Schwarzian of a composition of maps is governed by the *cocycle relation*:

$$S(f \circ g) = g^* S(f) + S(g) \tag{6.1}$$

In [OS], Osgood and Stowe construct a generalization of the Schwarzian derivative that acts on a pair of conformally equivalent Riemannian metrics on a manifold. We describe this generalization only in the case of conformal metrics on a Riemann surface, as this is the case we will use.

Given conformal metrics $\rho_1, \rho_2 \in L^2(S_{\frac{1}{2},\frac{1}{2}}(X))$, define

$$\sigma(\rho_1, \rho_2) = \log(\rho_2/\rho_1). \tag{6.2}$$

Then the *Schwarzian tensor $\beta(\rho_1, \rho_2)$ of $\rho_2$ relative to $\rho_1$* is defined as

$$\beta(\rho_1, \rho_2) = [\mathrm{Hess}_{\rho_1}(\sigma) - d\sigma \otimes d\sigma]^{2,0} \tag{6.3}$$

where we have written $\sigma$ instead of $\sigma(\rho_1, \rho_2)$ for brevity. This definition differs from that of Osgood and Stowe in that we take only the $(2, 0)$ part, whereas they consider the traceless part, which in this case is the sum of

$\beta$ and its complex conjugate[1]. For Riemann surfaces, the definition above seems more natural.

Using this definition, we can compute the Schwarzian tensor in local coordinates for a pair of conformal metrics:

$$\beta(\rho_1, \rho_2) = \left[(\sigma_2 - \sigma_1)_{zz} - {(\sigma_2)_z}^2 + {(\sigma_1)_z}^2\right] dz^2 \quad \text{where} \quad \rho_i = e^{\sigma_i}|dz| \tag{6.4}$$

The Schwarzian tensor generalizes $S(f)$ in the following sense: If $\Omega \subset \mathbb{C}$ and $\rho$ is the pullback of the Euclidean metric $|dz|^2$ of $\mathbb{C}$ under a holomorphic map $f : \Omega \to \mathbb{C}$, then

$$\beta(|dz|, \rho) = \beta(|dz|, f^*(|dz|)) = \frac{1}{2}S(f). \tag{6.5}$$

Generalizing the cocycle property of the Schwarzian derivative, the Schwarzian tensors associated to a triple of conformal metrics $(\rho_1, \rho_2, \rho_3)$ satisfy

$$\beta(\rho_1, \rho_3) = \beta(\rho_1, \rho_2) + \beta(\rho_2, \rho_3). \tag{6.6}$$

Note that $\beta(\rho, \rho) = 0$ for any conformal metric $\rho$, so we also have the antisymmetry relationship:

$$\beta(\rho_1, \rho_2) = -\beta(\rho_2, \rho_1) \tag{6.7}$$

Finally, the Schwarzian tensor is functorial with respect to conformal maps, i.e.

$$\beta(f^*(\rho_1), f^*(\rho_2)) = f^*(\beta(\rho_1, \rho_2)), \tag{6.8}$$

where $f$ is a conformal map between domains on Riemann surfaces, and $\rho_1, \rho_2$ are conformal metrics on the target of $f$.

For a domain $\Omega \subset \mathbb{C}$, we will say a conformal metric is *Möbius flat* if its Schwarzian tensor relative to the Euclidean metric vanishes. By the cocycle formula, this property is invariant under pullback by Möbius transformations.

**Lemma 6.1** (Osgood and Stowe [OS]). *After pulling back by a Möbius transformation and multiplying by a positive constant, a Möbius flat metric on a domain $\Omega \subset \mathbb{C}$ can be transformed to the restriction of exactly one of the following examples:*

(i) *The standard Euclidean metric of $\mathbb{C}$,*
(ii) *The spherical metric of $\mathbb{CP}^1 \simeq S^2 \subset \mathbb{R}^3$,*
(iii) *The hyperbolic metric on a round disk $D \subset \mathbb{CP}^1$,*

We call (i)-(iii) the *Euclidean*, *spherical*, and *hyperbolic* cases, respectively. By (6.6), the property of being Möbius flat is also equivalent to having vanishing Schwarzian tensor relative to any other Möbius flat metric.

While all Möbius-flat metrics on domains in $\hat{\mathbb{C}}$ have constant curvature, the converse is not true. In fact, a conformal metric has constant curvature if and only if its Schwarzian relative to a Möbius-flat metric is *holomorphic*

[1]We also use a slightly different notation than [OS]; we write $\beta(\rho_1, \rho_2)$ for the $(2, 0)$ part of what Osgood and Stowe call $B_{\rho_1^2}(\log(\rho_2/\rho_1))$.

[CDO1, Thm. 1]. For example, we can calculate the Schwarzian tensor of the (unique up to scale) complete Euclidean metric $|z^{-1}dz|$ on $\mathbb{C}^*$ relative to the Euclidean metric of $\mathbb{C}$:

$$\beta(|dz|, |z^{-1}\,dz|) = \frac{1}{4}\frac{dz^2}{z^2} \tag{6.9}$$

Note that everything except the constant $\frac{1}{4}$ in (6.9) can be derived by symmetry considerations; the constant itself is determined by calculation.

## 7. Decomposition of the Schwarzian

In this section we show that the Schwarzian derivative of the developing map of a grafted surface can be understood geometrically in terms of the grafting lamination using the Schwarzian tensor of Osgood and Stowe. While we restrict attention here to $\lambda \in \mathscr{ML}(S)$ supported on a union of simple closed geodesics, this condition will be eliminated in the proof of the main theorem by a continuity argument. We begin with a brief discussion of the motivation.

When the grafting lamination is a simple closed hyperbolic geodesic, the restriction of the developing map to the grafted part has a simple form: it is the composition of a uniformizing map $\widetilde{A} \to \mathbb{H}$ on the (universal cover of) the grafting cylinder $A$ and a map of the form $z \mapsto z^\alpha$, where $\alpha$ is determined by the measure on the geodesic. Since the Schwarzian derivative of a univalent map is bounded (by a theorem of Nehari), one can use the cocycle property (6.1) to determine the Schwarzian derivative of the developing map up to a bounded error. However, this approach gives a bound that depends on the homotopy class of the closed geodesic in an essential way, and offers little hope of an extension to more general measured laminations.

Rather than expressing part of the developing map as a composition, the generalized cocycle property (6.6) of the Schwarzian *tensor* suggests that we look for a conformal metric on the entire surface $X$ that interpolates between the hyperbolic metric and the pullback of a spherical metric on $\hat{\mathbb{C}}$ by the developing map. It turns out that the Thurston metric has the right properties to give a uniform estimate (as we will see in §11).

**Theorem 7.1** (Schwarzian decomposition). *Let $X(\lambda) \in P(X)$ be the projective structure on $X$ with grafting lamination $\lambda \in \mathscr{ML}(S)$, and suppose that $\lambda$ is supported on a union of simple closed geodesics. Let $\phi_T(\lambda) \in Q(X)$ be the Schwarzian derivative of its developing map, and $\Phi(\lambda)$ the grafting differential (which is not holomorphic). Then*

$$2\phi_T(\lambda) = 4\beta(\rho_0, \rho_\lambda) - \Phi(\lambda).$$

*Proof.* As the argument is essentially local, we suppress the distinction between metrics and differentials on $X$ and their lifts to equivariant objects on $\widetilde{X}$.

Let $\rho_{\hat{\mathbb{C}}}$ be a Möbius-flat metric on $\hat{\mathbb{C}}$ (e.g. a spherical metric). Using (6.5) we have

$$\phi_T(\lambda) = 2\beta(\rho_0, f^*\rho_{\hat{\mathbb{C}}})$$

where $f : \widetilde{X} \to \hat{\mathbb{C}}$ is the developing map. By the cocycle property of the Schwarzian tensor,

$$\beta(\rho_0, f^*\rho_{\hat{\mathbb{C}}}) = \beta(\rho_0, \rho_\lambda) + \beta(\rho_\lambda, f^*\rho_{\hat{\mathbb{C}}}).$$

Since the grafting differential is defined as $\Phi(\lambda) = 4\Phi(\kappa)$, it suffices to show

$$\Phi(\kappa) = -\beta(\rho_\lambda, f^*\rho_{\hat{\mathbb{C}}}) \tag{7.1}$$

almost everywhere on $X$, where $\kappa : X \to Y_\lambda = \operatorname{pr}_\lambda X$ is the collapsing map.

To prove (7.1), recall that when $\lambda$ is supported on a union of simple closed geodesics, the collapsing map and Thurston metric are based on two local models [Tan1, §2]:

(1) In the hyperbolic part $X_{-1}$, the collapsing map is an isometry, and

$$\Phi(\kappa)|_{X_{-1}} = 0.$$

On the other hand, the Thurston metric is the pullback by the developing map of the hyperbolic metric on a round disk in $\hat{\mathbb{C}}$. Thus in a neighborhood of a point in the hyperbolic part,

$$\beta(\rho_\lambda, f^*\rho_{\hat{\mathbb{C}}}) = \beta(f^*\rho_D, f^*\rho_{\hat{\mathbb{C}}}) \ = f^*\beta(\rho_D, \rho_{\hat{\mathbb{C}}}) = 0$$

where $\rho_D$ is the hyperbolic metric on a round disk $D \subset \hat{\mathbb{C}}$. Here we have used the naturality property (6.8) of $\beta$ and the fact that both $\rho_D$ and $\rho_{\hat{\mathbb{C}}}$ are Möbius flat.

(2) In the Euclidean (grafted) part $X_0$, which is a union of cylinders, the collapsing map is locally modeled on the projection of $\mathbb{H}$ to $i\mathbb{R}$ by $z \mapsto i|z|$, which has Hopf differential

$$\Phi(\kappa) = \frac{1}{4}\frac{dz^2}{z^2}.$$

In the same coordinates, the Thurston metric is the pullback of the cylindrical metric $|dz|/|z|$ on $\mathbb{C}^*$, so near a point in the grafted part,

$$\begin{aligned}\beta(\rho_\lambda, f^*\rho_{\hat{\mathbb{C}}}) &= \beta(f^*\rho_{\mathbb{C}^*}, f^*\rho_{\hat{\mathbb{C}}}) = f^*\beta(\rho_{\mathbb{C}^*}, \rho_{\hat{\mathbb{C}}}) \\ &= -f^*\beta(\rho_{\hat{\mathbb{C}}}, \rho_{\mathbb{C}^*}) = -\frac{1}{4}\frac{dz^2}{z^2}\end{aligned}$$

where in the last line we have used (6.9) and the fact that $\rho_{\hat{\mathbb{C}}}$ is Möbius flat.

Thus $\Phi(\kappa : X \to Y_\lambda)$ and $-\beta(\rho_\lambda, f^*\rho_{\hat{\mathbb{C}}})$ are equal in $X_0$ and $X_{-1}$, hence a.e. on $X$, which is (7.1), and the theorem follows. $\boxed{\text{Theorem 7.1}}$

In light of Theorem 7.1, the remaining obstacle to a geometric understanding of the Schwarzian derivative of the developing map is the Schwarzian tensor $\beta(\rho_0, \rho_\lambda)$ of the Thurston metric relative to the hyperbolic metric. After studying the Thurston metric in more detail, we will determine a bound for its Schwarzian tensor in §11.

## 8. NPC conformal metrics

Some of the properties of the Thurston metric on a grafted surface that we will use in the proof of the main theorem can be attributed to the fact that it is nonpositively curved (NPC). We devote this section and the next to a separate discussion of such metrics and the additional regularity properties they enjoy compared to general conformal metrics on a Riemann surface.

Consider a geodesic metric space $(M, d)$, i.e. a metric space in which the distance $d(x, y)$ is then length of some path joining $x$ and $y$. We say $(M, d)$ is nonpositively curved (NPC) space if all of its geodesic triangles are "thinner" than triangles in the plane with the same edge lengths (see [ABN] for details on this and equivalent definitions). A space with this property is also called CAT(0). The definition of an NPC space actually implies that it is simply connected, but we will also say that a metric manifold $(M, d)$ is NPC if the triangle condition is satisfied in its universal cover $(\widetilde{M}, \widetilde{d})$.

An NPC metric on a surface $S$ naturally induces a conformal structure:

**Theorem 8.1.** *Let $S$ be a compact surface and $d(\cdot, \cdot)$ an NPC metric compatible with the topology of $S$. Then there is a unique Riemann surface $X \in \mathscr{T}(S)$ and a conformal metric $\rho$ on $X$ inducing $d$, i.e. such that*

$$d(x, y) = \inf \left\{ \int_\gamma \rho(z) \, |dz| \;\middle|\; \gamma : ([0, 1], 0, 1) \to (X, x, y) \right\}.$$

Theorem 8.1 is essentially due to Rešetnyak (see [Reš1], [Reš2], and the recent survey [Reš3]) who shows that for an NPC metric on a two-dimensional manifold there is a local conformal homeomorphism to the disk $\mathbb{D} \subset \mathbb{C}$. Here "conformal" must be interpreted in terms of the preservation of angles between curves, which are defined in an NPC space using the distance function. A theorem of Huber implies that under these circumstances there is a conformal metric on the disk that gives $\rho$ as above [Hub].

Another proof of Theorem 8.1 is given in [Mes1] using the Korevaar-Schoen theory of harmonic maps to metric spaces; here the metric $\rho$ is obtained as the pullback metric tensor for a conformal harmonic map from a domain in $\mathbb{C}$ to a domain in the NPC surface $(S, d)$. This analysis leads naturally to more detailed regularity and nondegeneracy properties of $\rho$:

**Theorem 8.2** (Mese [Mes1])**.** *There is a one-to-one correspondence between NPC metrics $d(\cdot, \cdot)$ on $S$ and the distance functions arising from pairs $(X, \rho)$ with $X \in \mathscr{T}(S)$ and $\rho$ a conformal metric on $X$ such that $\rho \in W^{1,2}_{\mathrm{loc}}(X)$ and $\log \rho(z)$ is weakly subharmonic. Furthermore, if $\rho$ is such a conformal metric, then $\rho(z) > 0$ almost everywhere.*

The subharmonicity of $\log \rho(z)$ reflects the condition of nonpositive curvature; indeed if $\rho$ is a smooth, nondegenerate conformal metric then its

Gaussian curvature at a point $z$ is

$$K_\rho(z) = -\frac{\Delta \log \rho(z)}{\rho(z)^2} \tag{8.1}$$

and therefore

$$K_\rho(z) \leq 0 \iff \Delta \log \rho(z) \geq 0.$$

For more general NPC conformal metrics on $X$, there is no direct analogue of the Gaussian curvature function, but there is a curvature measure $\Omega_\rho$ with local expression

$$\Omega_\rho = -\Delta \log \rho \tag{8.2}$$

By approximation one can show that the Gauss-Bonnet theorem holds in this context, i.e. the total measure of $\Omega_\rho$ is $2\pi\chi(S)$.

So finally we define $\mathrm{NPC}(X)$ to be the set of conformal metrics $\rho \in L^2(S_{\frac{1}{2},\frac{1}{2}}(X))$ such that the induced distance function $d_\rho(\cdot,\cdot)$ makes $X$ into an NPC metric space. From the preceding discussion, $\rho(z)$ is then $W^{1,2}_{\mathrm{loc}}$, almost everywhere positive, $\log \rho(z)$ is subharmonic in a conformal coordinate chart, and the area $dA_\rho$ and curvature measures $\Omega_\rho$ are finite. Conversely, any NPC metric on $S$ whose associated conformal structure (as in Theorem 8.1) is $X$ gives rise to such a conformal metric.

## 9. Regularity and compactness for NPC metrics

We now compare the various NPC metrics on a fixed compact Riemann surface $X$. Let $\mathrm{NPC}_a(X) \subset \mathrm{NPC}(X)$ denote the set of NPC metrics on $X$ with total area $2\pi|\chi|a$ (i.e. $a$ times the hyperbolic area of $X$). For any $t > 0$, we have $\rho \in \mathrm{NPC}_a(X)$ if and only if $t\rho \in \mathrm{NPC}_{t^2a}(X)$, so we may as well consider only metrics of a fixed area, e.g. $\rho \in \mathrm{NPC}_1(X)$.

**Theorem 9.1** (Mese [Mes2, Thm. 29])**.** *The set of distance functions $\{d_\rho | \rho \in \mathrm{NPC}_1(X)\}$ is compact in the topology of uniform convergence. In particular, it is closed: a uniform limit of such distance functions is the distance function of an NPC metric on $X$.*

*Remark.* In [Mes2], Mese establishes this compactness result in the greater generality of metrics with a positive upper bound on curvature (i.e. $\mathrm{CAT}(k)$, rather than $\mathrm{CAT}(0)$) using the theory of harmonic maps to such metric spaces.

Conformal metrics $\rho_1$ and $\rho_2$ with uniformly close distance functions can differ significantly on a small scale; for example, such an estimate does not give any control on the modulus of continuity of the map of metric spaces $\mathrm{Id} : (X, \rho_1) \to (X, \rho_2)$. Since this is precisely the kind of control we will need for application to grafting, we now investigate the local properties of NPC metrics. Since it is smooth and uniquely determined, the hyperbolic metric $\rho_0 \in \mathrm{NPC}_1(X)$ is a good basis for comparison of regularity of NPC metrics on $X$.

**Theorem 9.2.** *For $X \in \mathscr{T}(S)$ with hyperbolic metric $\rho_0 \in \mathrm{NPC}_1(X)$ we have:*

(i) *For all $\rho \in \mathrm{NPC}_1(X)$, the map $\mathrm{Id} : (X, \rho_0) \to (X, \rho)$ is Lipschitz with constant depending only on $X$, i.e.*

$$\|\rho/\rho_0\|_\infty \leq C(X).$$

(ii) *For all $\rho_1, \rho_2 \in \mathrm{NPC}_1(X)$, the map $\mathrm{Id} : (X, \rho_1) \to (X, \rho_2)$ is bi-Hölder, with exponent and an upper bound on the Hölder norm depending only on $X$.*

*Remark.* Modulo the values of exponents and Lipschitz and Hölder norms, Theorem 9.2 is the best kind of estimate that could hold for a general NPC metric of fixed area: When $\rho = |\phi|^{\frac{1}{2}}$ for a holomorphic quadratic differential $\phi$, the map $(X, \rho) \to (X, \rho_0)$ is Hölder but not Lipschitz near the zeros of $\phi$.

*Proof (of Theorem 9.2).*
(i) Consider the map $\mathrm{Id} : (X, \rho_0) \to (X, \rho)$, which is conformal and thus harmonic. Its energy is the area of the image, $2\pi|\chi|$, thus by Korevaar and Schoen's regularity theorem for harmonic maps to NPC spaces, the Lipschitz constant of the map is bounded above by a constant that depends only on the hyperbolic metric $\rho_0$, i.e. on the conformal structure $X$ [KS].

(ii) In light of (i), it suffices to give a lower bound on the $(X, \rho)$-distance in terms of the $(X, \rho_0)$-distance for all $\rho \in \mathrm{NPC}_1(X)$. That is, for $x, y \in X$ sufficiently close, we must show that

$$d_\rho(x, y) \geq C d_{\rho_0}(x, y)^K \tag{9.1}$$

for some $C, K > 0$ that depend on $X$.

To establish (9.1), we will use the fact that $\log(\rho)$ is subharmonic (by the NPC condition), which limits the size of the set where $\log(\rho)$ is large and negative (making $\rho$ close to zero). The distance estimate comes from an effective form of the classical fact that the set where a subharmonic function takes the value $-\infty$ has zero Hausdorff dimension; this will prevent $\rho$ from being too small on a significant fraction of a geodesic segment.

First we make the problem local by covering $X$ with disks where the metric $\rho$ is everywhere bounded above and somewhere bounded below. Specifically, we cover $X$ by $N$ hyperbolic disks of a fixed radius $R$, each parameterized by the unit disk $\mathbb{D} \subset \mathbb{C}$. Here $N$ and $R$ depend only on $X$. We can also find $r < 1$ depending on $X$ such that the images of $\mathbb{D}_r = \{|z| < r\}$ under the $N$ chart maps still cover $X$.

For such a covering by disks, there are constants $M > m > -\infty$, also depending only on $X$, such that on each such disk the conformal density $\rho(z)$ satisfies

$$\begin{gathered} \log \rho(z) < M \text{ for all } z \in \mathbb{D} \\ \log \rho(z_0) > m \text{ for some } z_0 \in \mathbb{D}_r \end{gathered} \tag{9.2}$$

The existence of the upper bound $M$ follows from the Lipschitz estimate in part (i) above. As for $m$, if such a bound did not exist, one could find a sequence $\rho_i \in \mathrm{NPC}_1(X)$ converging uniformly to 0 on $\mathbb{D}_r$. In particular the associated distance functions $d_{\rho_i}$ could not accumulate on the distance function of any NPC metric $d_\infty$, contradicting Theorem 9.1.

We now show that for a conformal density function $\rho$ of an NPC metric on the unit disk satisfying (9.2) and any $x, y \in \mathbb{D}_r$ sufficiently close,

$$d_\rho(x, y) \geq C|x - y|^K \tag{9.3}$$

where $C$ and $K$ depend only on $X$. Since the hyperbolic metric $\rho_0$ is smooth and comparable to the Euclidean metric (by constants depending on $X$ and the covering by disks), part (ii) of the theorem then follows with a different constant $C$.

Fix $r'$ such that $r < r' < 1$. For $x, y \in \mathbb{D}_r$ close enough, there is a minimizing $\rho$-geodesic segment $\gamma \subset \mathbb{D}_{r'}$ joining them, because by Theorem 9.1 the $\rho$-distance from $\partial\mathbb{D}_r$ to $\partial\mathbb{D}_{r'}$ is bounded below for all $\rho \in \mathrm{NPC}_1(X)$.

We now use a result of Brudnyi on the level sets of subharmonic functions:

**Theorem 9.3** (Brudnyi [Bru, Prop. 1])**.** *Let $u$ be a subharmonic function on $\mathbb{D}$ such that*

$$\sup_{\mathbb{D}} u < M \quad \text{and} \quad \sup_{\mathbb{D}_r} u > m$$

*for some $r < 1$ and $-\infty < m < M < \infty$. Then for each $\epsilon > 0$ and $d > 0$ there is a countable family of disks $\mathbb{D}(z_i, r_i)$ with*

$$\sum_i r_i^d < \frac{(2\epsilon)^d}{d}$$

*and such that*

$$u(z) \geq m - C(M - m)\left(1 + \log\left(\frac{1}{\epsilon}\right)\right)$$

*for all $z \in \mathbb{D}_r$ outside these disks. Here $C = C(r) > 0$ depends only on $r$.*

Applying Theorem 9.3, there is a collection of disks in $\mathbb{D}$ with radii $r_i$ satisfying

$$\sum_i r_i \leq \frac{|x - y|}{2}$$

and such that on the complement of these disks in $\mathbb{D}_{r'}$, the conformal factor $\log\rho(z)$ satisfies

$$\log\rho(z) > m - C(M - m)\left(1 + \log\frac{4}{|x - y|}\right) \tag{9.4}$$

Since the path $\gamma$ connects $x$ and $y$, the exclusion of these disks leaves a subset $\gamma_0 \subset \gamma$ of Euclidean length at least $|x - y|/2$ where the bound (9.4)

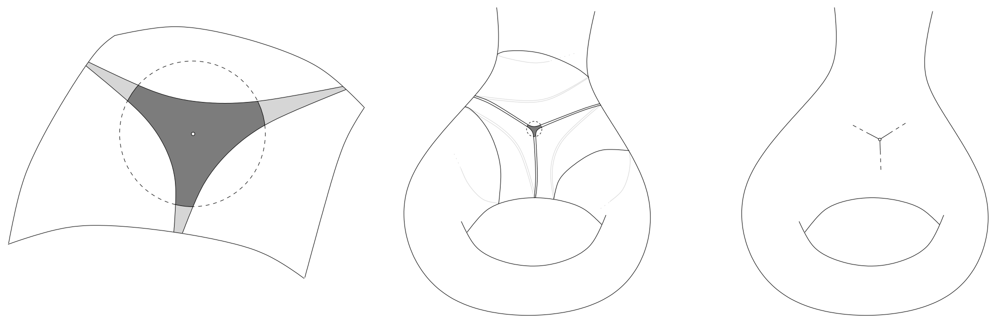

**Figure 2:** *Scaling the Thurston metric (left) to have unit area makes its curvature concentrate near a finite set of points (center) in a manner reminiscent of a quadratic differential metric (right).*

is satisfied. In particular,

$$\begin{aligned} d_\rho(x,y) &= \int_\gamma \rho(z)\,|dz| \geq \int_{\gamma_0} \rho(z)\,|dz| \\ &\geq \exp\left(\inf_{\gamma_0} \log \rho(z)\right) \int_{\gamma_0} |dz| \\ &\geq \exp\left(C_1 - C_2 \log(4/|x-y|)\right) \frac{|x-y|}{2} \\ &\geq C_3\,|x-y|^K \end{aligned} \tag{9.5}$$

where $C_i$ are unspecified constants that depend only on $X$. This is the desired bound from (9.3), so we have proved (ii).

$\boxed{\text{Theorem 9.2}}$

## 10. Curvature of the Thurston metric

We now begin to study the geometry of the Thurston metric. The idea is that for large grafting laminations $\lambda$, the Thurston metric looks a lot like the singular flat metric coming from a holomorphic quadratic differential. This is because the $\rho_\lambda$-area of the hyperbolic part $X_{-1}(\lambda)$ is fixed (and equal to the hyperbolic area of $Y_\lambda$), while that of the Euclidean part $X_0(\lambda)$ grows with $\lambda$. In fact, we have (cf. [Tan1, §3])

$$\mathrm{Area}(X_0(\lambda), \rho_\lambda) = \ell(\lambda, Y_\lambda),$$

while by Theorem 5.1,

$$\ell(\lambda, Y_\lambda) \approx E(\lambda, X) \to \infty \text{ as } \lambda \to \infty,$$

where $E(\lambda, X)$ is the extremal length of $\lambda$ on $X$. Thus if we rescale the Thurston metric to have constant area, all of its curvature is concentrated in a very small part of the surface, which is reminiscent of the conical singularities of a quadratic differential metric (see Figure 2).

Since the area of $(X, \rho_\lambda)$ is approximately $E(\lambda, X)$ (i.e. up to a bounded additive constant), the ratio $\rho_\lambda/\rho_0$ is approximately $E(\lambda, X)^{\frac{1}{2}}$ in an average sense. Since $\rho_\lambda/\mathrm{Area}(X, \rho_\lambda)^{\frac{1}{2}} \in \mathrm{NPC}_1(X)$, part (i) of Theorem 9.2 provides an upper bound of the same order, i.e.

$$\frac{\rho_\lambda(x)}{\rho_0(x)} \leq C(X)\left(1 + E(\lambda, X)^{\frac{1}{2}}\right) \tag{10.1}$$

The next theorem quantifies the sense in which the curvature of $\rho_\lambda$ becomes concentrated for large $\lambda$: A finite set of small hyperbolic disks on $X$ suffices to cover all but an exponentially small part of $X_{-1}(\lambda)$, measured with respect to the background metric $\rho_0$.

**Theorem 10.1** (curvature concentration). *For any $\epsilon > 0$, $X \in \mathscr{T}(S)$ and $\lambda \in \mathscr{ML}(S)$, let $X_{-1}(\lambda)$ denote the subset of $X$ where the Thurston metric $\rho_\lambda$ is hyperbolic. Then there are $N = 2|\chi|$ points $x_1, \ldots, x_N \in X$ such that*

$$\mathrm{Area}\,(X_{-1}(\lambda) - B, \rho_0) \leq C \exp(-\alpha E(\lambda, X)^{\frac{1}{2}})),$$

*where $B = \bigcup_i B_\epsilon(x_i)$, $B_\epsilon(x)$ is the hyperbolic ball of radius $\epsilon$ centered at $x$, and the constants $C$ and $\alpha$ depend on $\epsilon$ and $X$ (but not on $\lambda$).*

The curvature concentration phenomenon described by Theorem 10.1 arises from a simple geometric property of ideal triangles.

**Lemma 10.2** (ideal triangles). *Let $T \subset \mathbb{H}^2$ be a hyperbolic ideal triangle (a region bounded by three pairwise asymptotic geodesics), and $x \in T$. Then*

$$\mathrm{Area}(T - B_r(x)) \leq Ce^{-r}$$

*where $C$ depends only on $d(x, \partial T)$.*

*Proof.* This is an exercise in hyperbolic geometry; an explicit calculation shows that this is true for the center of $T$, and then the full statement follows since for each $\epsilon > 0$, $(T - N_\epsilon(\partial T))$ is compact. $\boxed{\text{Lemma 10.2}}$

*Proof of Theorem 10.1.* Fix $\epsilon > 0$. Every geodesic lamination on a hyperbolic surface can be enlarged (non-uniquely) to a geodesic lamination whose complement is a union of $N = 2|\chi|$ hyperbolic ideal triangles. Given $\lambda \in \mathscr{ML}(S)$, let $\tau$ be such an enlargement of the supporting geodesic lamination of $\lambda$ on the hyperbolic surface $Y_\lambda = \mathrm{pr}_\lambda X$.

Let $T_1, \ldots, T_N$ be the ideal triangles comprising $Y_\lambda - \tau$, and $t_1, \ldots, t_N \in Y_\lambda$ points in the thick parts of these triangles (e.g. their centers). Since $X = \mathrm{gr}_\lambda Y_\lambda$, each of the triangles $T_i$ naturally includes into $X$, and this inclusion is an isometry for the Thurston metric. Let $x_i \in X$ denote the point corresponding to $t_i \in T_i$ by this inclusion. The images of $T_i$ in $X$ cover all of $X_{-1}(\lambda)$ except a null set consisting of the finitely many $\rho_\lambda$-geodesics added to $\mathrm{supp}(\lambda)$ to obtain $\tau$.

Since $\rho_\lambda/\mathrm{Area}(X, \rho_\lambda)^{\frac{1}{2}} \in \mathrm{NPC}_1(X)$, part (ii) of Theorem 9.2 implies that the metrics $\rho_0$ and $\rho_\lambda/\mathrm{Area}(X, \rho_\lambda)^{\frac{1}{2}}$ are Hölder equivalent. Thus for some $k > 0$, the hyperbolic disk $B_\epsilon(x_i)$ contains a $\rho_\lambda$-disk of radius at least

$C(X)E(\lambda,X)^{\frac{1}{2}}\epsilon^k$, where $C$ and $k$ depend only on $X$. Here we have used the fact that $\mathrm{Area}(X,\rho_\lambda) \approx E(\lambda,X)$ and $\mathrm{Area}(X,\rho_\lambda)$ is bounded below (by $\mathrm{Area}(X,\rho_0) = 2\pi|\chi|$).

Applying Lemma 10.2 to $T_i$, we see that $B_\epsilon(x_i)$ covers all of the image of $T_i$ in $X$ except for a region of $\rho_\lambda$-area at most $C_1(X)\exp(-C_2(X)E(\lambda,X)^{\frac{1}{2}}\epsilon^k)$. Since $\rho_\lambda \geq \rho_0$, the same upper bound holds for the hyperbolic area, and applying this to each of the $N$ triangles we obtain:

$$\begin{aligned}\mathrm{Area}(X_{-1}(\lambda) - B, \rho_0) &\leq \mathrm{Area}(X_{-1}(\lambda) - B, \rho_\lambda)\\ &= \mathrm{Area}\left(\bigcup_i T_i - \bigcup_i B_\epsilon(x_i), \rho_\lambda\right)\\ &\leq NC_1(X)\exp(-C_2(X)E(\lambda,X)^{\frac{1}{2}}\epsilon^k)\end{aligned}$$

Taking $\alpha = C_2(X)\epsilon^k$, Theorem 10.1 follows.

$\boxed{\text{Theorem 10.1}}$

## 11. The Schwarzian tensor of the Thurston metric

In this section we find an upper bound for the norm of the Schwarzian tensor of the Thurston metric restricted to a subset of $X$. As in Theorem 10.1, we could take this subset to be the complement of finitely many small hyperbolic balls, but it is technically simpler to work with a fixed domain, so we use:

**Lemma 11.1.** *For any $X \in \mathscr{T}(S)$ there exist $\epsilon_0(X), \delta(X) > 0$ such that if $B \subset X$ is the union of $N = 2|\chi|$ hyperbolic balls of radius $\epsilon \leq \epsilon_0(X)$, then $(X - B)$ contains an embedded hyperbolic ball of radius $\delta(X)$.*

*Proof.* Choose $\epsilon_0$ small enough so that $2\epsilon_0$ is less than the hyperbolic injectivity radius of $X$ and so that $2N$ balls of radius $2\epsilon_0$ cannot cover $X$ (say, by area considerations). Then for any union of $N$ balls of radius $\epsilon < \epsilon_0$, there is a point in $X$ whose distance from all of them is at least $\epsilon_0$. Thus $(X - B)$ contains a hyperbolic ball $D$ of radius $\epsilon_0$, which is necessarily embedded, for any $B$ as in the statement of the Lemma. We set $\delta = \epsilon_0$. $\boxed{\text{Lemma 11.1}}$

Now we continue our study of the Thurston metric with a series of analytic results that apply Theorem 10.1 (curvature concentration). Recall from (6.2) that $\sigma(\rho_1,\rho_2) = \log(\rho_2/\rho_1)$.

**Lemma 11.2.** *Fix $X \in \mathscr{T}(S)$ and $p < \infty$. For each $\lambda \in \mathscr{ML}(S)$ there is a hyperbolic ball $D \subset X$ of radius $\delta(X)$ such that*

$$\|\Delta_{\rho_0}\sigma(\rho_0,\rho_\lambda)\|_{L^p(D,\rho_0)} < C(p,X).$$

*Proof.* Using the formula for the Gaussian curvature of a conformal metric (8.1), we have

$$\Delta_{\rho_0}\sigma(\rho_\lambda,\rho_0) = -\frac{1}{\rho_0^2}\Delta\log\rho_0 + \frac{1}{\rho_0^2}\Delta\log\rho_\lambda = K_{\rho_0} - \frac{\rho_\lambda^2}{\rho_0^2}K_{\rho_\lambda}.$$

Note that the Gaussian curvature of the Thurston metric exists almost everywhere because its conformal factor is $C^{1,1}$ [KP]. Since $K_{\rho_0} \equiv -1$ and the $\rho_0$-area of $X$ is $2\pi|\chi|$ we have

$$\|K_{\rho_0}\|_{L^p(X,\rho_0)} = (2\pi|\chi|)^{\frac{1}{p}} = C(p)$$

So we need only establish an $L^p$ bound for the second term. From (10.1), we have $\rho_\lambda^2/\rho_0^2 \leq C\,(1 + E(\lambda, X))$. On the other hand, $|K_{\rho_\lambda}| \leq 1$, and this function is supported in $X_{-1}(\lambda)$. By Theorem 10.1 we have

$$\mathrm{Area}(X_{-1}(\lambda) - B, \rho_0) \leq C \exp\left(-\alpha E(\lambda, X)^{\frac{1}{2}}\right)$$

Where $C$ and $\alpha$ depend on $X$ and $\epsilon$. Combining these estimates, we find

$$\begin{aligned}\|\frac{\rho_\lambda^2}{\rho_0^2} K_{\rho_\lambda}\|_{L^p(X-B,\rho_0)} &\leq \left(\sup \frac{\rho_\lambda^2}{\rho_0^2}\,|K_{\rho_\lambda}|\right)(\mathrm{Area}\,(\mathrm{supp}\,K_{\rho_\lambda} \cap (X - B), \rho_0))^{\frac{1}{p}} \\ &\leq C\,(1 + E(\lambda, X)) \exp\left(-\alpha p^{-1} E(\lambda, X)^{\frac{1}{2}}\right).\end{aligned}$$

In particular, $\|\frac{\rho_\lambda^2}{\rho_0^2} K_{\rho_\lambda}\|_{L^p(X-B,\rho_0)} \to 0$ as $E(\lambda, X) \to \infty$, and we have a uniform upper bound on the norm depending only on $X$, $\epsilon$, and $p$.

Taking $\epsilon = \epsilon_0(X)$ and applying Lemma 11.1, the set $(X - B)$ contains a hyperbolic ball $D$ of radius $\delta(X)$, and restriction to this set only decreases the norm. So on $D$ we obtain an upper bound depending on $p$ and $X$. $\boxed{\text{Lemma 11.2}}$

In what follows it will be convenient to normalize the area of the Thurston metric; let $\hat{\rho}_\lambda$ denote the positive multiple of the Thurston metric with the same area as the hyperbolic metric $\rho_0$ on $X$, i.e.

$$\hat{\rho}_\lambda = \left(\frac{2\pi|\chi|}{\mathrm{Area}(X, \rho_\lambda)}\right)^{\frac{1}{2}} \rho_\lambda.$$

**Lemma 11.3.** *Fix $X \in \mathscr{T}(S)$ and $p < \infty$. For each $\lambda \in \mathscr{ML}(S)$ there is a hyperbolic ball $D_{\delta/2} \subset X$ of radius $\delta(X)/2$ such that*

$$\|\sigma(\rho_0, \hat{\rho}_\lambda)\|_{L^p(D_{\delta/2})} < C(p, X).$$

*Remark.* Using $\hat{\rho}_\lambda$ instead of $\rho_\lambda$ only changes $\sigma(\rho_0, \rho) = \log(\rho/\rho_0)$ by a constant, so Lemma 11.3 says that $\sigma(\rho_0, \rho_\lambda)$ is close to a constant in an $L^p$ sense.

*Proof.* Since $\hat{\rho}_\lambda \in \mathrm{NPC}_1(X)$, part (i) of Theorem 9.2 implies that $\sigma(\rho_0, \hat{\rho}_\lambda) < C_0(X)$. We will use the bound on the curvature of $\hat{\rho}_\lambda$ to turn this global upper bound into an estimate of the $L^p$ norm in a small disk.

Let $F = C_0 - \sigma(\rho_0, \hat{\rho}_\lambda)$, so $F$ is a nonnegative function, and let $D = D_\delta$ be the hyperbolic ball of radius $\delta$ provided by Lemma 11.2. Then we have

$$\|\Delta_{\rho_0} F\|_{L^p(D_\delta)} = \|\Delta_{\rho_0} \sigma(\rho_0, \hat{\rho}_\lambda)\|_{L^p(D_\delta)} \leq C_1(p, X).$$

Let $D_r$ denote the hyperbolic ball of radius $r$ concentric with $D$. We now want to show that $F$ cannot be large throughout $D_{\delta/4}$, or equivalently, show that there is a point $z \in D_{\delta/4}$ where $\hat{\rho}_\lambda = \exp(C_0 - F)\rho_0$ is not too close to zero. But as we saw in the proof of Theorem 9.2, this follows from the compactness of distance functions for metrics in $\mathrm{NPC}_1(X)$ (Theorem 9.1): if $\hat{\rho}_\lambda$ could be arbitrarily small throughout a disk of definite hyperbolic radius (in this case, $\delta(X)/4$), a limiting argument would give a metric in $\mathrm{NPC}_1(X)$ that vanishes on an open set, a contradiction. Thus $\inf_{D_{\delta/4}} F \leq C_2(X)$.

Combining these bounds on $F$ and $\Delta F$, we apply the weak Harnack inequality (see [GT, Thm. 8.18]) to obtain

$$\begin{aligned} \|F\|_{L^p(D_{\delta/2})} &\leq C_3(\delta(X)) \left( \|\Delta_{\rho_0} F\|_{L^p(D_\delta)} + \inf_{D_{\delta/4}} F \right) \\ &\leq C_3(\delta(X)) \left(C_1(p,X) + C_2(X)\right) \\ &= C(p,X) \end{aligned} \tag{11.1}$$

Since $F = C_0 - \sigma(\rho_0, \hat{\rho}_\lambda)$, the Lemma follows by algebra. $\boxed{\text{Lemma 11.3}}$

Combining the preceding lemmas and standard elliptic theory, we can now bound the Schwarzian tensor:

**Theorem 11.4.** *Fix $X \in \mathscr{T}(S)$. For each $\lambda \in \mathscr{ML}(S)$ there is a hyperbolic ball $D_{\delta/4} \subset X$ of radius $\delta(X)/4$ such that*

$$\|\beta(\rho_0, \rho_\lambda)\|_{L^1(D_{\delta/4})} \leq C(X).$$

*Proof.* Let $D_{\delta/2}$ be as in Lemma 11.3 and let $D_{\delta/4} \subset D_{\delta/2}$ be the concentric ball of radius $\delta(X)/4$. By standard elliptic theory (e.g. [GT, Thm. 9.11]) we have the Sobolev norm estimate

$$\|u\|_{W^{2,2}(D_{\delta/4},\rho_0)} \leq C(\delta) \left( \|\Delta_{\rho_0} u\|_{L^2(D_{\delta/2},\rho_0)} + \|u\|_{L^2(D_{\delta/2},\rho_0)} \right),$$

that is, the second derivatives of $u$ are bounded in terms of $u$ and its Laplacian.

Applying this to $u = \sigma(\rho_0, \hat{\rho}_\lambda)$, the terms on the right hand side are bounded by Lemmas 11.2 and 11.3, respectively, giving

$$\|\sigma(\rho_0, \hat{\rho}_\lambda)\|_{W^{2,2}(D_{\delta/4},\rho_0)} \leq C_1(X). \tag{11.2}$$

Since the $W^{2,2}$ norm bounds both the derivative of $\sigma(\rho_\lambda, \rho_0)$ in $L^2$ and its Hessian in $L^2$ (hence also $L^1$, by the Cauchy-Schwartz inequality), the definition of the Schwarzian tensor (6.3) gives

$$\|\beta(\rho_\lambda, \rho_0)\|_{L^1(D_{\delta/4},\rho_0)} < C_2(X) \|\sigma(\rho_0, \hat{\rho}_\lambda)\|_{W^{2,2}(D_{\delta/4},\rho_0)},$$

which together with (11.2) gives the desired bound. $\boxed{\text{Theorem 11.4}}$

## 12. Proof of the main theorem

Now that we have an estimate on the Schwarzian tensor of the Thurston metric (Theorem 11.4) and the decomposition of the Schwarzian derivative of a $\mathbb{CP}^1$ structure (Theorem 7.1), the proof of the main theorem is straightforward. We will need a lemma about holomorphic quadratic differentials in order to extend a bound on a small hyperbolic ball to one on $X$.

**Lemma 12.1.** *For any $X \in \mathscr{T}(S)$ and $\delta > 0$ there is a constant $C(\delta, X)$ such that*
$$\|\psi\|_{L^1(X)} \leq C(\delta, X)\|\psi\|_{L^1(D_\delta)}$$
*for all $\psi \in Q(X)$ and any hyperbolic ball $D_\delta \subset X$ of radius $\delta$.*

*Proof.* The inequality is homogeneous, so we need only prove it for $\psi \in Q(X)$ with $\|\psi\|_{L^1(X)} = 1$. The set of all unit-norm quadratic differentials is compact and equicontinuous.

Suppose on the contrary that there is no such constant $C(\delta, X)$. Then there is a sequence $\psi_n \in Q(X)$ and $\delta$-balls $D_n \subset X$ such that $\|\psi_n\|_{L^1(X)} = 1$ and $\|\psi_n\|_{L^1(D_n)} \to 0$ as $n \to \infty$. Taking a subsequence we can assume $\psi_n \to \psi_\infty$ uniformly and $D_n \to D_\infty$, where we say a sequence of $\delta$-balls converges if their centers converge. By uniform convergence of $\psi_n$ we obtain
$$\|\psi_\infty\|_{L^1(D_\infty)} = \lim_{n\to\infty} \|\psi_n\|_{L^1(D_n)} = 0.$$
But $\psi_\infty$ is a nonzero holomorphic quadratic differential, which vanishes at only finitely many points, so this is a contradiction. $\boxed{\text{Lemma 12.1}}$

*Proof (of Theorem 1.1).* First suppose that $\lambda$ is supported on a union of simple closed geodesics.

Comparing both $\phi_F(\lambda)$ and $\phi_T(\lambda)$ to the grafting differential $\Phi(\lambda)$, and applying Theorem 7.1, we have
$$\begin{aligned} 2\phi_T(\lambda) + \phi_F(\lambda) &= \Big(2\phi_T(\lambda) + \Phi(\lambda)\Big) \; - \; \Big(\Phi(\lambda) - \phi_F(\lambda)\Big) \\ &= 4\beta(\rho_\lambda, \rho_0) - \Big(\Phi(\lambda) - \phi_F(\lambda)\Big) \end{aligned}$$
Taking the $L^1$ norm, we apply Theorem 11.4 to the first term and Theorem 5.2 to the second, giving
$$\|2\phi_T(\lambda) + \phi_F(\lambda)\|_{L^1(D_{\delta/4})} \leq 4C_1(X) + C_2\left(1 + E(\lambda, X)^{\frac{1}{2}}\right) \tag{12.1}$$
where $D_{\delta/4}$ is a hyperbolic ball of radius $\delta(X)/4$. Note that Theorem 5.2 bounds the $L^1(X)$ norm, so the same upper bound applies to the $L^1(D_{\delta/4})$ norm used here.

Since $(2\phi_T(\lambda)+\phi_F(\lambda))$ is holomorphic, we apply Lemma 12.1 to the norm bound (12.1), and obtain
$$\|2\phi_T(\lambda) + \phi_F(\lambda)\|_{L^1(X)} \leq C_3(X)\left(1 + E(\lambda, X)^{\frac{1}{2}}\right),$$
where $C_3(X)$ incorporates both the constants $C_1$ and $C_2$ from (12.1) and $C(\delta(X)/4, X)$ from Lemma 12.1. Since $\|\phi_F(\lambda)\|_{L^1(X)} = E(\lambda, X)$, this proves Theorem 1.1 for $\lambda$ supported on closed geodesics.

Finally, since both $\phi_T$ and $\phi_F$ are homeomorphisms, the function $\lambda \mapsto \|2\phi_T(\lambda) + \phi_F(\lambda)\|_1$ is continuous on $\mathscr{ML}(S)$. Since we have established a bound for this function on the dense subset of $\mathscr{ML}(S)$ consisting of weighted simple closed geodesics, the same bound applies to all laminations. $\boxed{\text{Theorem 1.1}}$

Note that the harmonic maps estimate (Theorem 5.2) provides the dominant factor in the upper bound of Theorem 1.1, and the proof shows that any improvement to this estimate would give a corresponding improvement to the bound on $\|2\phi_T(\lambda) + \phi_F(\lambda)\|_1$:

**Corollary 12.2** (of proof)**.** *Fix $X \in \mathscr{T}(S)$. For all $\lambda \in \mathscr{ML}(S)$ we have*

$$\|2\phi_T(\lambda) + \phi_F(\lambda)\|_{L^1(X)} \leq C(X) \left(1 + \|\Phi(\lambda) - \phi_F(\lambda)\|_{L^1(X)}\right)$$

It seems natural to ask whether it is possible to make the bound completely independent of $\lambda$, that is:

*Question.* Is there a constant $C(X)$ such that $\|2\phi_T(\lambda) + \phi_F(\lambda)\|_{L^1(X)} < C(X)$ for all $\lambda \in \mathscr{ML}(S)$?

Even if the bound can be made independent of $\lambda$, the dependence on $X$ is probably necessary. If there is a bound that depends only on the topological type, one would need to select the right norm on $Q(X)$ (perhaps using something other than $L^1$). While all of the standard norms on $Q(X)$ (e.g. $L^1$, $L^p$, $L^\infty$, etc.) are equivalent when $X$ is fixed, the constants are not uniform as $X \to \infty$.

## 13. Applications: Holonomy and Fuchsian $\mathbb{CP}^1$ structures

We now turn to an application of Theorem 1.1 in the study of $\mathbb{CP}^1$ structures and their holonomy representations. Some background on this topic is necessary before we state the results.

Recall that a projective surface $Z \in \mathscr{P}(S)$ has a holonomy representation $\eta(Z) : \pi_1(S) \to \mathrm{PSL}_2(\mathbb{C})$, which is unique up to conjugation (see §4). The association of a holonomy representation to a projective structure defines a map

$$\eta : \mathscr{P}(S) \to \mathscr{V}(S)$$

where $\mathscr{V}(S)$ is the $\mathrm{PSL}_2(\mathbb{C})$ character variety of $\pi_1(S)$, i.e.

$$\mathscr{V}(S) = \mathrm{Hom}(\pi_1(S), \mathrm{PSL}_2(\mathbb{C})) /\!\!/ \, \mathrm{PSL}_2(\mathbb{C})$$

and $\mathrm{PSL}_2(\mathbb{C})$ acts on the set of homomorphisms by conjugation. While this quotient (in the sense of geometric invariant theory) is only defined up to birational equivalence, in this case one can construct a good representative embedded in $\mathbb{C}^n$ using trace functions (see [CS]). Then $\eta$ maps to the smooth points of this variety and is a holomorphic local homeomorphism, though it is not proper (and in particular is not a covering map). The range of $\eta$ has been studied by a number of authors; recently, it was shown that $\eta$ is essentially surjective onto a connected component of $\mathscr{V}(S)$ [GKM], proving a conjecture of Gunning [Gun2].

We will be interested in the holonomy of $\mathbb{CP}^1$ structures on a fixed Riemann surface $X$, i.e. the restriction of $\eta$ to $P(X)$. The resulting holomorphic immersion

$$\eta_X : P(X) \to \mathscr{V}(S)$$

is proper [GKM, §11.4] [Tan2] and injective [Kra].

Within $\mathscr{V}(S)$ there is the closed set $\mathrm{AH}(S)$ of discrete and faithful representations, and its interior $\mathscr{QF}(S)$, which consists of quasi-Fuchsian representations. The quasi-Fuchsian representations are exactly those whose limit sets are quasi-circles in $\mathbb{CP}^1$; similarly, the set $\mathscr{F}(S)$ of Fuchsian representations consists of those whose limit sets are round circles.

Let $K(X)$ denote the set of projective structures on $X$ with discrete holonomy representations. Shiga and Tanigawa showed that the interior $\operatorname{Int} K(X)$ is exactly the set of projective structures on $X$ with quasi-Fuchsian holonomy, i.e.

$$\operatorname{Int} K(X) = \eta_X^{-1}(\mathscr{QF}(S)).$$

There is a natural decomposition of $\operatorname{Int} K(X)$ into countably many open and closed subsets according to the topology of the corresponding developing maps; the components in this decomposition are naturally indexed by integral measured laminations (or *multicurves*) $\gamma \in \mathscr{ML}_{\mathbb{Z}}(S)$:

$$\operatorname{Int} K(X) = \bigsqcup_{\gamma \in \mathscr{ML}_{\mathbb{Z}}(S)} B_\gamma(X)$$

The developing maps of projective structures in $B_\gamma(X)$ have "wrapping" behavior described by the lamination $\gamma$ (see [Kap, Ch. 7] [Gol]).

The set $B_0(X)$ corresponding to the empty multicurve $0 \in \mathscr{ML}_{\mathbb{Z}}(S)$ is the *Bers slice* of $X$, i.e. the set of projective structures on $X$ that arise from the domains of discontinuity of quasi-Fuchsian representations of $\pi_1(S)$ [Shi]. When identified with a subset of $Q(X)$ using the Schwarzian derivative, $B_0(X)$ is the connected component of $\operatorname{Int} K(X)$ containing the origin, and is a bounded, contractible open set. The set $B_0(X)$ can also be characterized as the set of projective structures on $X$ with quasi-Fuchsian holonomy and injective developing maps.

We call the other sets $B_\gamma(X)$ (with $\gamma \neq 0$) the *exotic Bers slices*, because they are natural analogues of the Bers slice $B_0(X)$, but they consist of quasi-Fuchsian projective structures whose developing maps are not injective (which are called "exotic" $\mathbb{CP}^1$-structures). Compared to $B_0(X)$, little is known about the exotic Bers slices $B_\gamma(X)$; in particular it is not known whether they are connected or bounded.

Each exotic Bers slice $B_\gamma(X)$ contains a distinguished point $c_\gamma = c_\gamma(X)$, the *Fuchsian center*, which is the unique $\mathbb{CP}^1$ structure in $B_\gamma(X)$ with Fuchsian holonomy. While in general the connection between the multicurve $\gamma$ and the grafting laminations of projective structures in $B_\gamma(X)$ is difficult to determine, for $c_\gamma$ the grafting lamination is just $2\pi\gamma$ (see [Gol]), i.e.

$$c_\gamma = \phi_T(2\pi\gamma) \in B_\gamma(X).$$

The application of Theorem 1.1 we have in mind involves the distribution of the Fuchsian centers within $P(X) \simeq Q(X)$. Associated to a multicurve $\gamma$

there is a *Jenkins-Strebel differential* $s_\gamma$,

$$s_\gamma = \phi_F(2\pi\gamma) \in Q(X)$$

which is a holomorphic quadratic differential whose noncritical horizontal trajectories are closed and homotopic to the curves in the support of $\gamma$. Note that $-s_\gamma$ is then a differential with closed vertical trajectories, so the sign in the definition is a matter of convention.

Because the grafting laminations of the Fuchsian centers are integral measured laminations (up to the factor $2\pi$), Theorem 1.1 implies that the associated points in $Q(X)$ are close to differentials with closed trajectories. Specifically, we have:

**Theorem 13.1.** *For any compact Riemann surface $X$ (with underlying smooth surface $S$) and all multicurves $\gamma \in \mathscr{ML}_{\mathbb{Z}}(S)$, the Fuchsian center $c_\gamma \in P(X) \simeq Q(X)$ and the $2\pi$-integral Jenkins-Strebel differential $s_\gamma \in Q(X)$ satisfy*

$$\|2c_\gamma + s_\gamma\|_1 \leq C(X)\left(1 + \sqrt{\|s_\gamma\|_1}\right)$$

*where $C(X)$ is a constant depending only on $X$.*

*Proof.* Using the definition of $c_\gamma$ and $s_\gamma$, this is immediate from Theorem 1.1:

$$\|2c_\gamma + s_\gamma\|_1 = \|2\phi_T(2\pi\gamma) + \phi_F(2\pi\gamma)\|_1 \leq C(X)\left(1 + \sqrt{\|\phi_F(2\pi\gamma)\|_1}\right)$$

$\boxed{\text{Theorem 13.1}}$

Theorem 13.1 implies that there is a rich structure to the exotic Bers slices $B_\gamma(X)$ within the vector space $Q(X)$ of holomorphic quadratic differentials; for example, each rational ray $\mathbb{R}^+ \cdot s_\gamma \subset Q(X)$, $\gamma \in \mathscr{ML}_Z(S)$ approximates the positions of an infinite sequence of Fuchsian centers and their surrounding "islands" of quasi-Fuchsian holonomy (though the distance from this line to the centers may itself go to infinity, but at a slower rate).

For example, if we look at a sequence $\{n\gamma \mid n = 1, 2, \ldots\} \subset \mathscr{ML}_Z(S)$, then the norm $\|s_{n\gamma}\|_1 = 4\pi^2 n^2 E(\gamma, X)$ grows quadratically with $n$, while by Theorem 13.1 we have $\|2c_{n\gamma} + s_{n\gamma}\|_1 = O(n)$ as $n \to \infty$.

**Numerical examples.** In some cases the positions of the Fuchsian centers in $Q(X)$ can be computed numerically. While such experiments do not yield new theoretical results, we present them here to illustrate the connection between Fuchsian centers and Jenkins-Strebel differentials (as in Theorem 13.1) and the associated geometry of the holonomy map.

While the preceding discussion involved only compact surfaces, the analogous theory of punctured surfaces with *bounded* $\mathbb{CP}^1$ structures (those which are represented by meromorphic quadratic differentials having at most simple poles at the punctures) is more amenable to computation. We will focus on the case where $X$ is a punctured torus, so $P(X)$ is a one-dimensional complex affine space. Here $X$ is commensurable with a planar Riemann

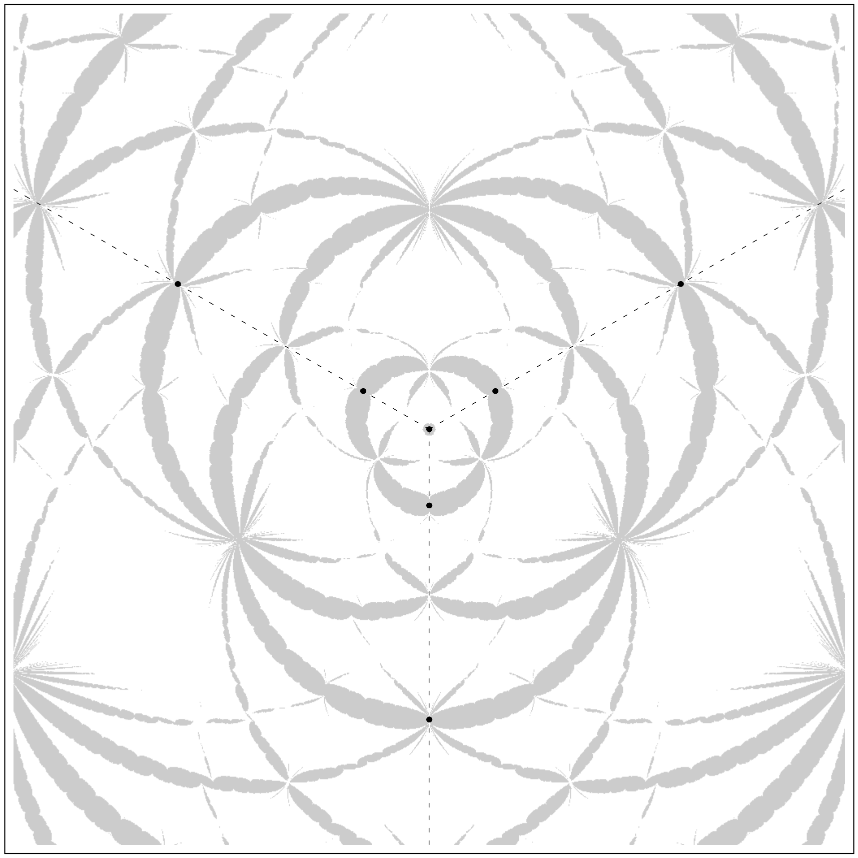

**Figure 3:** *Fuchsian centers for the hexagonal punctured torus, corresponding to the empty multicurve (the center) and the first two multiples of the three systoles.*

surface (a four-times-punctured $\mathbb{CP}^1$), so it is possible to compute the holonomy representation of a $\mathbb{CP}^1$ structure on $X$ by numerical integration of an ODE around contours in $\mathbb{C}$. Existing discreteness algorithms for punctured torus groups can then be used to create pictures of Bers slices, the discreteness locus $K(X)$, and of the positions of the Fuchsian centers within islands of quasi-Fuchsian holonomy. The images that follow were created using a computer software package implementing these techniques [D3]. See [KoSu] [KSWY] for further discussion of numerical methods.

Figure 3 shows part of $P(X)$ where $X$ is the hexagonal punctured torus, i.e. the result of identifying opposite edges of a regular hexagon in $\mathbb{C}$ and removing one cycle of vertices. The image is centered on the Bers slice, and regions corresponding to projective structures with discrete holonomy have been shaded. Each of the seven Fuchsian centers in this region of $P(X)$ is marked; these correspond to the empty multicurve and the three shortest curves on the hexagonal torus with multiplicities one and two.

The dashed curves in Figure 3 are the *pleating rays* in $P(X)$ consisting of projective structures with grafting laminations $\{t\gamma | t \in \mathbb{R}^+\}$, where $\gamma$ is one of the three systoles. Our use of the term "pleating ray" is somewhat different than that of Keen-Series and others, as we mean that the projective class of a bending lamination is fixed for a family of equivariant pleated planes in $\mathbb{H}^3$, whereas in [KeSe] and elsewhere it is often assumed that the associated $\mathrm{PSL}_2(\mathbb{C})$ representation is quasi-Fuchsian and that the pleated surface is one of its convex hull boundary surfaces. The pleating rays in $P(X)$ (as we have defined them) naturally interpolate between the Fuchsian centers, which appear at the $2\pi$-integral points.

The hexagonal torus is a special case because it has many symmetries. In fact, the pleating rays for the systoles are forced (by symmetry) to be Euclidean rays emanating from the origin in the directions of the associated Jenkins-Strebel differentials. Thus, while we expect (based on Theorem 1.1 for compact surfaces) that the Fuchsian centers associated to multiples of the systoles lie near the lines of Jenkins-Strebel differentials, in this example the Fuchsian centers lie on the lines, which coincide with the pleating rays.

Figure 4 shows part of $P(X)$ for a punctured torus $X$ without symmetries. Five Fuchsian centers are marked, corresponding to a certain simple closed curve $\gamma$ with multiplicity $n$, $0 \leq n \leq 4$. While there is no longer an intrinsic symmetry that forces the centers to lie on a straight line in $P(X) \simeq \mathbb{C}$, the pleating ray that contains these centers (the dashed curve) is almost indistinguishable from the line of Jenkins-Strebel differentials at the resolution of the figure. In fact, a sector of angle approximately $4 \times 10^{-5}$ centered on the Jenkins-Strebel ray appears to contain the part of the pleating ray visible in the figure.

Figure 5 shows the norm of $2\phi_T + \phi_F$ along the pleating ray from Figure 4. In this region, this norm is everywhere less than 2, while the first Fuchsian center $c_\gamma$ has norm greater than 14, and the width of the region in Figure 4 is approximately 250. This should be contrasted with the distance between the pleating ray and the line of Jenkins-Strebel differentials as unparameterized curves, which is much smaller still. Nevertheless, the numerical results suggest that the Fuchsian centers are not exactly collinear, with ratios of neighboring centers in Figure 4 apparently having small but nonzero imaginary parts.

## 14. Application: Compactification of $P(X)$ and $\mathscr{P}(S)$

In this section we describe another application of Theorem 1.1 that extends the results of [D1] on $\mathbb{CP}^1$ structures and the asymptotics of grafting and pruning.

As described in §4, Thurston's projective extension of grafting provides a homeomorphism between the space $\mathscr{P}(S)$ of $\mathbb{CP}^1$ structures on the differentiable compact surface $S$ and the product $\mathscr{ML}(S) \times \mathscr{T}(S)$:

$$\mathscr{ML}(S) \times \mathscr{T}(S) \xrightarrow{\mathrm{Gr}} \mathscr{P}(S).$$

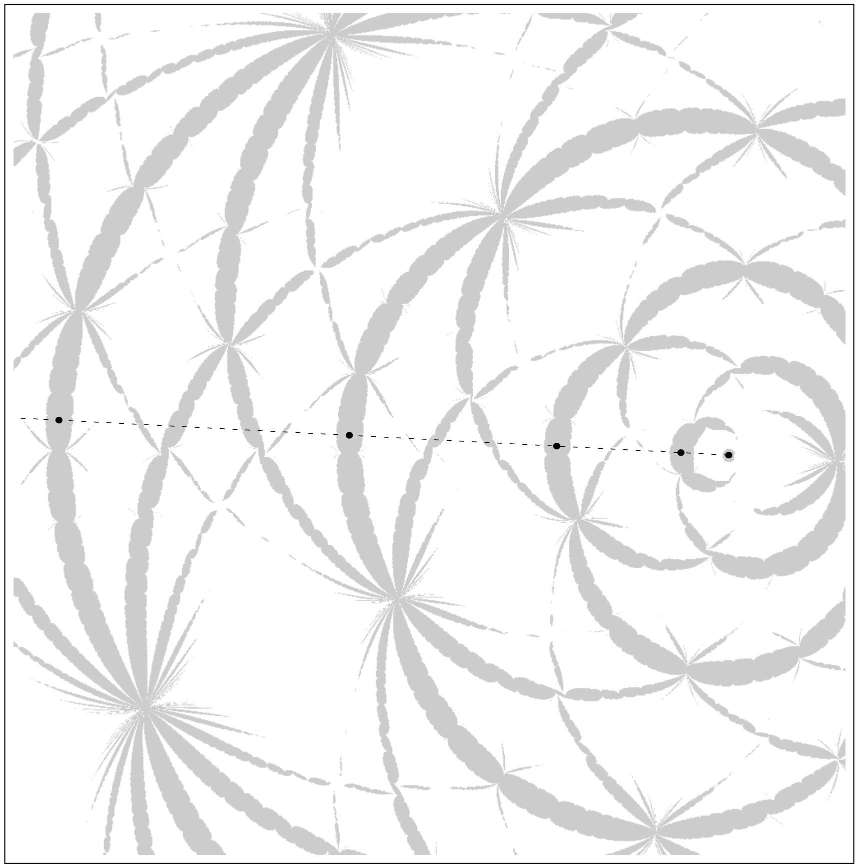

**Figure 4:** *Fuchsian centers for a punctured torus with no symmetries; the modular parameter of the underlying Euclidean torus is approximately* $\tau = 0.369 + 1.573i$. *The right-most marked point is the origin, which is the Fuchsian center in the Bers slice. The other Fuchsian centers correspond to multiples of a simple closed curve* $\gamma$, *which lie on the* $\gamma$*-pleating ray* $\{\phi_T(t\gamma) \mid t \in \mathbb{R}^+\}$ *(the dashed curve). The* $L^1$ *distance between opposite sides of the image is approximately* 250.

Thus the set $P(X) \subset \mathscr{P}(S)$ of $\mathbb{CP}^1$ structures on a fixed Riemann surface $X$ corresponds, using grafting, to a set of pairs $M_X = \{(\lambda, Y) \in \mathscr{ML}(S) \times \mathscr{T}(S) \mid \mathrm{gr}_\lambda Y = X\}$.

In [D1], it is shown that the image of $P(X)$ in $\mathscr{ML}(S) \times \mathscr{T}(S)$ is well-behaved with respect to natural compactifications of the two factors–the Thurston compactification $\overline{\mathscr{T}(S)}$ and the projective compactification $\overline{\mathscr{ML}(S)} = \mathscr{ML}(S) \sqcup \mathbb{P}\mathscr{ML}(S)$. The asymptotic behavior of $P(X)$ is described in terms of the *antipodal involution* $i_X : \mathscr{ML}(S) \to \mathscr{ML}(S)$ (and its projectivization $i_X : \mathbb{P}\mathscr{ML}(S) \to \mathbb{P}\mathscr{ML}(S)$), where $i_X(\lambda) = \mu$ if and only if $\lambda$ and $\mu$ are measure-equivalent to the vertical and horizontal measured foliations of a holomorphic quadratic differential $\phi \in Q(X)$ [D1, §4]. Specifically, we have:

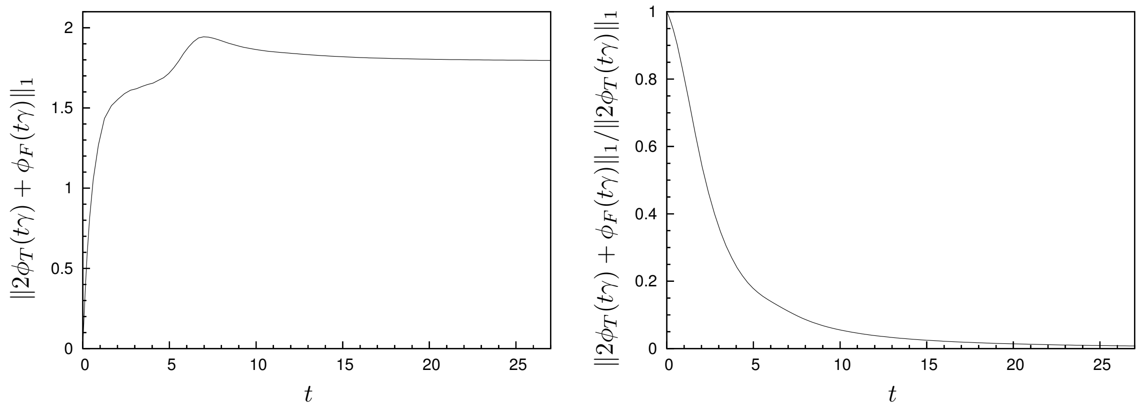


**Figure 5:** *The $L^1$ norm of $(2\phi_T(t\gamma) + \phi_F(t\gamma)) \in Q(X)$ as a function of $t$ (left), and as a fraction of the $L^1$ norm of $2\phi_T(t\gamma)$ (right). Here $X$ is the punctured torus with modular parameter $\tau = 0.369 + 1.573i$ and $\gamma$ is the curve on $X$ corresponding to the pleating ray in Figure 4.*

**Theorem 14.1** ([D1, Thm 1.2])**.** *For all $X \in \mathscr{T}(S)$, the boundary of $M_X = \{(\lambda, Y) \in \mathscr{ML}(S) \times \mathscr{T}(S) \mid \mathrm{gr}_\lambda Y = X\}$ in $\overline{\mathscr{ML}(S)} \times \overline{\mathscr{T}(S)}$ is the graph of the projectivized antipodal involution, i.e.*

$$\overline{M_X} = M_X \sqcup \Gamma(i_X)$$

*where*

$$\Gamma(i_X) = \{([\lambda], [i_X(\lambda)]) \mid \lambda \in \mathscr{ML}(S)\} \subset \mathbb{P}\mathscr{ML}(S) \times \mathbb{P}\mathscr{ML}(S).$$

A deficiency in this description of the image of $P(X)$ in $\mathscr{ML}(S) \times \mathscr{T}(S)$ is that it does not relate the boundary to the Poincaré parameterization of $P(X)$ using quadratic differentials.

On the other hand, there *is* a natural map from the boundary of the space of holomorphic quadratic differentials to $\mathbb{P}\mathscr{ML}(S) \times \mathbb{P}\mathscr{ML}(S)$: Consider the map

$$\mathscr{F}^\perp \times \mathscr{F} : Q(X) \to \mathscr{MF}(S) \times \mathscr{MF}(S)$$

which records the vertical and horizontal measured foliations of a holomorphic quadratic differential. Using the natural identification between $\mathscr{MF}(S)$ and $\mathscr{ML}(S)$, we also have a map $\Lambda^\perp \times \Lambda : Q(X) \to \mathscr{ML}(S) \times \mathscr{ML}(S)$. Because it is homogeneous, there is an induced map between projective spaces

$$\Lambda^\perp \times \Lambda : \mathbb{P}^+ Q(X) \to \mathbb{P}\mathscr{ML}(S) \times \mathbb{P}\mathscr{ML}(S),$$

where $\mathbb{P}^+Q(X) = (Q(X) - \{0\})/\mathbb{R}^+$. The image of this map is, by definition, the graph of the antipodal involution $i_X : \mathbb{P}\mathscr{ML}(S) \to \mathbb{P}\mathscr{ML}(S)$, and $\Lambda^\perp \times \Lambda$ intertwines the action of $-1$ on $\mathbb{P}^+Q(X)$ and the involution exchanging factors of $\mathbb{P}\mathscr{ML}(S) \times \mathbb{P}\mathscr{ML}(S)$.

Since $P(X)$ and $Q(X)$ are identified using the Schwarzian derivative, we can use the projective compactification $\overline{Q(X)} = Q(X) \sqcup \mathbb{P}^+Q(X)$ to obtain the *Schwarzian compactification* $\overline{P(X)}_S$; a sequence of projective structures converges in $\overline{P(X)}_S$ if their Schwarzian derivatives can be rescaled (by positive real factors) so as to converge in $(Q(X) - \{0\})$.

Using Theorem 1.1 we can extend Theorem 14.1 and show that the inclusion $P(X) \hookrightarrow \mathscr{ML}(S) \times \mathscr{T}(S)$ extends continuously to $\overline{P(X)}_S$ and has $\Lambda^\perp \times \Lambda$ as its boundary values:

**Theorem 14.2.** *For each $X \in \mathscr{T}(S)$, the inclusion $P(X) \hookrightarrow \mathscr{ML}(S) \times \mathscr{T}(S)$ obtained by taking Thurston's grafting coordinates for the projective structures on $X$ extends continuously to a map*

$$\overline{P(X)}_S \to \overline{\mathscr{ML}(S)} \times \overline{\mathscr{T}(S)}$$

*where $\overline{P(X)}_S = P(X) \sqcup \mathbb{P}^+Q(X)$ is the projective compactification using the Schwarzian derivative, $\overline{\mathscr{T}(S)}$ is the Thurston compactification, and $\overline{\mathscr{ML}(S)} = \mathscr{ML}(S) \sqcup \mathbb{P}\mathscr{ML}(S)$. When restricted to the boundary, this extension agrees with the map*

$$\Lambda^\perp \times \Lambda : \mathbb{P}^+Q(X) \to \mathbb{P}\mathscr{ML}(S) \times \mathbb{P}\mathscr{ML}(S)$$

*which sends a projective class of differentials to the projective measured laminations associated to its vertical and horizontal foliations. In particular, the boundary of $P(X)$ in $\overline{\mathscr{ML}(S)} \times \overline{\mathscr{T}(S)}$ is the graph of the antipodal involution $i_X : \mathbb{P}\mathscr{ML}(S) \to \mathbb{P}\mathscr{ML}(S)$.*

*Proof.* The idea is that the results of [D1] imply a similar extension statement for a compactification of $P(X)$ using harmonic maps, while Theorem 1.1 shows that this compactification is the same as the one obtained using the Schwarzian derivative (up to exchanging the vertical and horizontal foliations).

Consider a divergent sequence in $P(X)$, with associated grafting laminations $\lambda_i$; let $Y_i = \mathrm{pr}_{\lambda_i} X$. The proof of Theorem 14.1 in [D1] uses Wolf's theory of harmonic maps between Riemann surfaces and from Riemann surfaces to $\mathbb{R}$-trees (see [Wol2], [Wol3]) to show that if $h_i : X \to Y_i$ is the harmonic map compatible with the markings, and $\Phi_i \in Q(X)$ is its Hopf differential, then

$$\begin{aligned} \lim_{i\to\infty} [\Lambda(\Phi_i)] &= [\lambda] \in \mathbb{P}\mathscr{ML}(S) \\ \lim_{i\to\infty} [\Lambda^\perp(\Phi_i)] &= \lim_{i\to\infty} Y_i = [i_X(\lambda)] \in \mathbb{P}\mathscr{ML}(S). \end{aligned} \tag{14.1}$$

To rephrase these results, define the *harmonic maps compactification*

$$\overline{P(X)}_h = P(X) \sqcup \mathbb{P}^+Q(X)$$

where a sequence of projective structures converges to $[\Phi] \in \mathbb{P}^+Q(X)$ if the sequence $\Phi_i$ of Hopf differentials of the associated harmonic maps converges projectively to $\Phi$ (compare [Wol1]). Then (14.1) says that the inclusion

$P(X) \hookrightarrow \mathscr{ML}(S) \times \mathscr{T}(S)$ using the grafting coordinates extends continuously to the harmonic maps compactification,

$$\overline{P(X)}_h \hookrightarrow \overline{\mathscr{ML}(S)} \times \overline{\mathscr{T}(S)},$$

and that the boundary of this extension is

$$\Lambda \times \Lambda^\perp : \mathbb{P}^+Q(X) \to \mathbb{P}\mathscr{ML}(S) \times \mathbb{P}\mathscr{ML}(S).$$

By Theorem 1.1, the projective limit of the Schwarzian derivatives $\phi_T(\lambda_i)$ of a divergent sequence $X(\lambda_i)$ is the same as that of the sequence $-\phi_F(\lambda_i)$, where the sign is significant since $\mathbb{P}^+Q(X)$ is the set of rays, rather than lines, in $Q(X)$. On the other hand, by Theorem 5.2, the grafting differentials $\Phi(\lambda_i)$ and $\phi_F(\lambda_i)$ have the same projective limit, which is also the projective limit of the Hopf differentials of the harmonic maps $h_i : X \to Y_i$ by Theorem 9.1 of [D1] (see also [D2]).

As a result, the harmonic maps compactification and the Schwarzian compactification are asymptotically related by the projectivization of the linear map $-1 : \mathbb{P}^+Q(X) \to \mathbb{P}^+Q(X)$, which interchanges vertical and horizontal foliations. In particular the boundary map of $\overline{P(X)}_S$ exists and is given by $\Lambda^\perp \times \Lambda = (\Lambda \times \Lambda^\perp) \circ (-1)$. $\boxed{\text{Theorem 14.2}}$

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