GRAFTING, PRUNING, AND THE ANTIPODAL MAP ON MEASURED LAMINATIONS

DAVID DUMAS

Abstract. Grafting a measured lamination on a hyperbolic surface defines a self-map of Teichmüller space, which is a homeomorphism by a result of Scannell and Wolf. In this paper we study the large-scale behavior of pruning, which is the inverse of grafting.

Specifically, for each conformal structure $X \in \mathscr{T}(S)$, pruning X gives a map $\mathscr{ML}(S) \to \mathscr{T}(S)$. We show that this map extends to the Thurston compactification of $\mathscr{T}(S)$, and that its boundary values are the natural *antipodal involution* relative to X on the space of projective measured laminations.

We use this result to study Thurston's grafting coordinates on the space of $\mathbb{CP}^1$ structures on S. For each $X \in \mathscr{T}(S)$, we show that the boundary of the space $P(X)$ of $\mathbb{CP}^1$ structures on X in the compactification of the grafting coordinates is the graph $\Gamma(i_X)$ of the antipodal involution $i_X : \mathbb{P}\mathscr{ML}(S) \to \mathbb{P}\mathscr{ML}(S)$.

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1. Introduction

Grafting is a procedure that begins with a hyperbolic structure $Y \in \mathscr{T}(S)$ in the Teichmüller space of a surface S of negative Euler characteristic and a measured geodesic lamination $\lambda \in \mathscr{ML}(S)$. By replacing λ with a thickened version that carries a natural Euclidean metric, a new conformal structure $X = \mathrm{gr}_\lambda Y \in \mathscr{T}(S)$, the *grafting of* Y *along* λ, is obtained.

Scannell and Wolf have shown that for each lamination $\lambda \in \mathscr{ML}(S)$, the conformal grafting map $\mathrm{gr}_\lambda : \mathscr{T}(S) \to \mathscr{T}(S)$ is a homeomorphism, thus there is an inverse or *pruning map* $\mathrm{pr}_\lambda : \mathscr{T}(S) \to \mathscr{T}(S)$ [SW].

In this paper we describe the large-scale behavior of pruning $X \in \mathscr{T}(S)$ in terms of the conformal geometry of X. This description is based on the map

$$\Lambda : Q(X) \to \mathscr{ML}(S)$$

which records the measured lamination equivalent to the horizontal foliation of a holomorphic quadratic differential. Hubbard and Masur showed that Λ is a homeomorphism [HM], so we can use it to transport the involution $(\phi \mapsto -\phi)$ of $Q(X)$ to an involutive homeomorphism $i_X : \mathscr{ML}(S) \to \mathscr{ML}(S)$. Since the map Λ is homogeneous, i_X descends to an involution on $\mathbb{P}\mathscr{ML}(S) = (\mathscr{ML}(S) - \{0\})/\mathbb{R}^+$,

$$i_X : \mathbb{P}\mathscr{ML}(S) \to \mathbb{P}\mathscr{ML}(S),$$

which we call the *antipodal involution with respect to* X, since it is conjugate by Λ to the actual antipode map on the vector space $Q(X)$.

Statement of results. In §10 we show that i_X governs the large-scale behavior of the map $\mathscr{ML}(S) \to \mathscr{T}(S)$ which prunes X along a given lamination, i.e. $\lambda \mapsto \mathrm{pr}_\lambda X$. Specifically, let $\overline{\mathscr{ML}(S)}$ denote the natural compactification of $\mathscr{ML}(S)$ by $\mathbb{P}\mathscr{ML}(S)$, and let $\overline{\mathscr{T}(S)}$ denote the Thurston compactification of Teichmüller space, which also has boundary $\mathbb{P}\mathscr{ML}(S)$.

Theorem 1.1 (Antipodal limit). *The pruning map with basepoint* X*, written* $\lambda \mapsto \mathrm{pr}_\lambda X$*, extends continuously to a map* $\overline{\mathscr{ML}(S)} \to \overline{\mathscr{T}(S)}$ *whose boundary values are exactly the antipodal map* $i_X : \mathbb{P}\mathscr{ML}(S) \to \mathbb{P}\mathscr{ML}(S)$*.*

An equivalent formulation of Theorem 1.1 characterizes the fibers of the grafting map $\mathrm{gr} : \mathscr{ML}(S) \times \mathscr{T}(S) \to \mathscr{T}(S)$:

Theorem 1.2 (Fibers of grafting). *Let* $M_X = \mathrm{gr}^{-1}(X) \subset \mathscr{ML}(S) \times \mathscr{T}(S)$ *denote the set of all pairs* (λ, Y) *that graft to give* $X \in \mathscr{T}(S)$*. Then the boundary of* M_X *in* $\overline{\mathscr{ML}(S)} \times \overline{\mathscr{T}(S)}$ *is the graph of the antipodal involution, i.e.*

$$\overline{M_X} = M_X \sqcup \Gamma(i_X)$$

where

$$\Gamma(i_X) = \{([\lambda], [i_X(\lambda)]) \mid \lambda \in \mathscr{ML}(S)\} \subset \mathbb{P}\mathscr{ML}(S) \times \mathbb{P}\mathscr{ML}(S)$$

The equivalence of Theorems 1.1 and 1.2 follows from the definition of pruning (see §2).

Let $\mathscr{P}(S)$ denote the space of $\mathbb{CP}^1$ surfaces, i.e. surfaces with geometric structures modeled on $(\mathbb{CP}^1, \mathrm{PSL}_2(\mathbb{C}))$, and $P(X) \subset \mathscr{P}(S)$ the subset having underlying conformal structure X. Thurston's parameterization of $\mathbb{CP}^1$ structures is based on a homeomorphism

$$\mathscr{ML}(S) \times T(S) \xrightarrow{\sim} \mathscr{P}(S)$$

that is a generalization of grafting (see [KT]). This allows us to view $P(X)$ as a subset of $\mathscr{ML}(S) \times \mathscr{T}(S)$, and it is natural to try to understand the relationship between the conformal structure X and the pairs $(\lambda, Y) \subset \mathscr{ML}(S) \times \mathscr{T}(S)$ thus determined. Equivalently, it is natural to compare Thurston's geometric parameterization of $\mathbb{CP}^1$ surfaces via grafting with the classical complex-analytic viewpoint.

Theorem 1.2 gives some information about how these two perspectives on $\mathscr{P}(S)$ compare, as it provides an explicit description of the boundary of $\overline{P(X)}$ as a subset of $\overline{\mathscr{ML}(S)} \times \overline{\mathscr{T}(S)}$:

Theorem 1.3 (Boundary $P(X)$). *For each $X \in \mathscr{T}(S)$, the boundary of $P(X)$ is the graph of the antipodal involution i_X:*

$$\partial P(X) = \Gamma(i_X) \subset \partial(\overline{\mathscr{ML}(S)} \times \overline{\mathscr{T}(S)})$$

In particular, the closure of $P(X)$ is a ball of dimension $6g-6$, where g is the genus of S. Theorem 1.3 follows immediately from Theorem 1.2 because $P(X)$ corresponds to a fiber Thurston's $\mathbb{CP}^1$-structure generalization of grafting (see §2).

Collapsing and harmonic maps. Our main results are obtained by studying the *collapsing* map $\kappa : \mathrm{gr}_\lambda Y \to Y$ that collapses the grafted portion of the surface back to the geodesic lamination on Y.

The key observation that drives the proof of Theorem 1.1 is that the collapsing map is nearly harmonic, having energy exceeding that of the associated harmonic map by at most a constant depending only on the topology of S. This energy comparison is due to Tanigawa [Tan], and relies on an inequality of Minsky for length distortion of harmonic maps [Min].

Geometric limits. For the purposes of Theorem 1.1, which is a result about the asymptotic behavior of pruning, we are interested in pruning a fixed surface along a divergent sequence of laminations. For such a sequence, the energies of the associated collapsing maps also diverge, while remaining withing $O(1)$ of the energies of the homotopic harmonic maps. This allows us to promote the collapsing map to a genuine harmonic map by rescaling the hyperbolic metric of Y and taking a geometric limit. This limit construction relies on the compatibility between equivariant Gromov-Hausdorff convergence and harmonic maps from Riemann surfaces as established by Bestvina [Bes] and Paulin [Pau]; the more general theory of Korevaar and

Schoen (see [KS1], [KS2]) for harmonic maps to metric spaces could also be used.

The resulting limit harmonic map has values in an $\mathbb{R}$-tree, and there is a detailed structure theory for such harmonic maps due to Wolf ([W1], [W3], [W4]). It follows that a single holomorphic quadratic differential–the Hopf differential of the limit harmonic map–encodes both the Thurston limit of the pruned surfaces and the projective limit of the grafting laminations via its vertical and horizontal foliations. This gives rise to the antipodal relationship expressed by Theorem 1.1.

Asymmetry of Teichmüller geodesics. It is tempting to compare the role of the antipodal involution in Theorem 1.1 to that of the geodesic involution in a symmetric space. Indeed, the antipodal map relative to X exchanges projective measured laminations $([\lambda], [\mu])$ defining Teichmüller geodesics that pass through X, just as the geodesic involution exchanges endpoints (at infinity) of geodesics through a point in a symmetric space.

However, in an appendix we provide an example showing that this analogy does not work, because Teichmüller geodesics are badly behaved with respect to the Thurston compactification. Our example is based on properties of Teichmüller geodesic rays associated to Strebel differentials established by Masur (see [Mas1], [Mas2]). Specifically, we construct a pair of Teichmüller geodesics through a single point in $\mathscr{T}(S)$ that are asymptotic to each other in one direction, but which have distinct limit points in the other direction. This precludes the existence of any map of the Thurston boundary that plays the role of a Teichmüller geodesic involution.

Outline of the paper.

Section 2 introduces the main objects we study in this paper–the grafting, pruning, and collapsing maps. Using the definitions and basic properties of these objects, the equivalence of Theorems 1.1-1.3 is explained.

Sections 3-5 contain further background material needed for the proof of the main theorem. Specifically, Section 3 contains definitions and notation related to quadratic differentials (holomorphic and otherwise) and conformal metrics that are used in comparing the collapsing map to a harmonic map. The antipodal map on measured laminations is then introduced in Section 4 by means of the measured foliations attached to holomorphic quadratic differentials. Section 5 then describes the $\mathbb{R}$-trees dual to measured foliations and measured laminations that appear naturally when studying limits of harmonic maps.

Section 6 discusses results of Wolf (from [W2], [W3], [W4]) on harmonic maps between surfaces and $\mathbb{R}$-trees that allow us to characterize the extension of pruning in terms of the antipodal map on measured laminations.

Section 7 presents the key energy estimates that allow us to compare the collapsing map (and a dual object, the co-collapsing map) to an associated harmonic map. The energy of the collapsing map was first computed by

Tanigawa [Tan] in order to prove that grafting is proper; here we provide an analogous result on the properness of pruning.

Section 8 presents a complementary study of the Hopf differentials of the collapsing an co-collapsing maps. Here we establish a relationship between these (non-holomorphic) quadratic differentials and the grafting lamination that is analogous to the relationship between a holomorphic quadratic differential and its associated measured foliation. This relationship is essential in the proof of the main theorem.

Section 9 is dedicated to the proof of an analytic result on the convergence of nearly harmonic maps between surfaces to a genuinely harmonic map with values in an $\mathbb{R}$-tree. We take as inspiration the work of Wolf on harmonic maps to surfaces and $\mathbb{R}$-trees, extending his results for eventual application to the collapsing map of a grafted surface.

Section 10 completes the proof of Theorem 1.1 by considering the collapsing map of a divergent sequence of prunings, and applying the convergence result from §9. The same type of argument also allows us to bound the difference between the Hopf and Hubbard-Masur differentials associated to a grafted surface (Theorem 10.1), which can be seen as a finite analogue of the asymptotic statement in Theorem 1.1.

Appendix A contains a brief discussion of Teichmüller geodesics in relation to the main results of the paper. In particular we exhibit an asymmetry phenomenon that precludes the existence of an extension of the Teichmüller geodesic involution to the Thurston boundary of Teichmüller space.

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2. Grafting, pruning, and collapsing

Let S be a compact oriented surface of genus $g > 1$, and $\mathscr{T}(S)$ the Teichmüller space of marked conformal (equivalently, hyperbolic) structures on S. The simple closed hyperbolic geodesics on any hyperbolic surface $Y \in \mathscr{T}(S)$ are in one-to-one correspondence with the free homotopy classes of simple closed curves on S; therefore, when a particular hyperbolic metric is under consideration, we will use these objects interchangeably.

Fix $Y \in \mathscr{T}(S)$ and γ, a simple closed hyperbolic geodesic on Y. *Grafting* is the operation of removing γ from Y and replacing it with a Euclidean cylinder $\gamma \times [0, t]$, as shown in Figure 1. The resulting surface is called the grafting of Y along the weighted geodesic $t\gamma$, written $\mathrm{gr}_{t\gamma} Y$.

Associated to each grafted surface $\mathrm{gr}_{t\gamma} Y$ is a canonical map $\kappa : \mathrm{gr}_{t\gamma} Y \to Y$, the *collapsing map*, that collapses the grafted cylinder $\gamma \times [0, t]$ back onto the geodesic γ. There is also a natural C^1 conformal metric ρ_{Th} on $\mathrm{gr}_{t\gamma} Y$, the *Thurston metric*, that unites the hyperbolic metric on Y with the

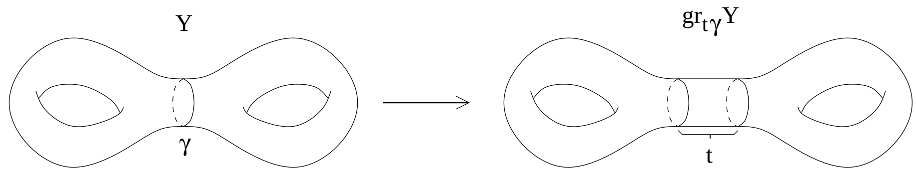

FIGURE 1. The basic example of grafting.

Euclidean metric of the cylinder $\gamma \times [0,t]$. The collapsing map is distance nonincreasing with respect to the hyperbolic metric on Y and the Thurston metric on $\mathrm{gr}_{t\gamma} Y$.

Grafting is compatible with the natural generalization of weighted simple closed geodesics to measured geodesic laminations. The space $\mathscr{ML}(S)$ of measured geodesic laminations is a contractible PL-manifold with an action of $\mathbb{R}^+$ in which the set of weighted simple closed hyperbolic geodesics for any hyperbolic metric $Y \in \mathscr{T}(S)$ forms a dense set of rays. Conversely, given $Y \in \mathscr{T}(S)$ each lamination $\lambda \in \mathscr{ML}(S)$ corresponds to a foliation of a closed subset of Y by complete nonintersecting hyperbolic geodesics equipped with a transverse measure of full support. For a detailed treatment of measured laminations, we refer the reader to [Thu].

Thurston has shown that grafting along simple closed curves extends continuously to arbitrary measured laminations, and thus defines a map

$$\mathrm{gr} : \mathscr{ML}(S) \times \mathscr{T}(S) \to \mathscr{T}(S) \text{ where } (\lambda, Y) \mapsto \mathrm{gr}_\lambda Y.$$

Morally, $\mathrm{gr}_\lambda Y$ is obtained from Y by thickening the geodesic realization of λ in a manner determined by the transverse measure.

As in the simple closed curve case, there is a collapsing map $\kappa : \mathrm{gr}_\lambda Y \to Y$ that collapses the grafted part $A \subset \mathrm{gr}_\lambda Y$ onto the geodesic realization of λ on Y, and a conformal metric ρ_{Th} on $\mathrm{gr}_\lambda Y$ that is hyperbolic on $\mathrm{gr}_\lambda Y - A$. These can be defined simply as limits of the corresponding objects for grafting along simple closed curves, or in terms of the geometry of the locally convex pleated plane in $\mathbb{H}^3$ associated to (Y, λ) [KT]. The metric ρ_{Th} is of class $C^{1,1}$ on $\mathrm{gr}_\lambda Y$, and thus its curvature is defined almost everywhere [KP].

Scannell and Wolf have shown that for each $\lambda \in \mathscr{ML}(S)$, the map $\mathrm{gr}_\lambda : \mathscr{T}(S) \to \mathscr{T}(S)$ is a homeomorphism [SW]. Thus there is the inverse, or *pruning* map

$$\mathrm{pr} : \mathscr{ML}(S) \times \mathscr{T}(S) \to \mathscr{T}(S) \text{ where } (\lambda, X) \mapsto \mathrm{pr}_\lambda X = \mathrm{gr}_\lambda^{-1} X.$$

In other words, for each $X \in \mathscr{T}(S)$ and $\lambda \in \mathscr{ML}(S)$, there is a unique way to present X as a grafting of some Riemann surface $Y = \mathrm{gr}_\lambda^{-1}(X)$ along λ, and pruning is the operation of recovering Y from the pair (λ, X).

We will be primarily interested in the pruning map when the surface X is fixed, i.e. the map $\lambda \mapsto \mathrm{pr}_\lambda X$ from $\mathscr{ML}(S)$ to $\mathscr{T}(S)$. Theorem 1.1 describes the asymptotic behavior of this map in terms of the conformal geometry of

X. Since the graph of this map is exactly the preimage of X under the grafting map, the continuous extension of the former to a homeomorphism of $\mathbb{P}\mathscr{ML}(S)$ also describes the closure of a fiber of $\mathrm{gr} : \mathscr{ML}(S) \times \mathscr{T}(S) \to \mathscr{T}(S)$; this is the content of Theorem 1.2.

Grafting and pruning are also related to $\mathbb{CP}^1$-structures on surfaces, i.e. geometric structures modeled on $(\mathbb{CP}^1, \mathrm{PSL}_2(\mathbb{C}))$. Each $\mathbb{CP}^1$ structure has an underlying conformal structure, and thus the deformation space $\mathscr{P}(S)$ of marked $\mathbb{CP}^1$-surfaces on S has a natural projection map $\pi : \mathscr{P}(S) \to \mathscr{T}(S)$ to Teichmüller space.

Thurston introduced an extension of grafting that produces complex projective structures (see [KT], [Tan]), and showed that the resulting map

$$\mathrm{Gr} : \mathscr{ML}(S) \times \mathscr{T}(S) \to \mathscr{P}(S)$$

is a homeomorphism. The composition of this projective grafting map with the projection π yields the conformal grafting map discussed above, i.e. $\pi \circ \mathrm{Gr} = \mathrm{gr}$. Thus the fiber $P(X) = \pi^{-1}(X) \subset \mathscr{P}(S)$ over $X \in \mathscr{T}(S)$ corresponds via Gr to the fiber of the conformal grafting map:

$$\mathrm{Gr}^{-1}(P(X)) = \mathrm{gr}^{-1}(X).$$

Applying this observation we see that Theorem 1.3 is a restatement of Theorem 1.2 in the language of $\mathbb{CP}^1$ geometry.

3. Conformal metrics and quadratic differentials

Our study of the pruning map will center on the collapsing map as a candidate for the harmonic map variational problem (i.e. minimizing energy in a given homotopy class). In this section we collect definitions and background material required for this variational analysis.

Fix a hyperbolic Riemann surface X and let $S(X)$ denote the space of measurable complex-valued quadratic forms on TX. A form $\beta \in S(X)$ can be decomposed according to the complex structure on X, i.e.

$$\beta = \beta^{2,0} + \beta^{1,1} + \beta^{0,2}$$

where $\beta^{i,j}$ is a symmetric form of type (i,j). Then $|\beta^{1,1}|$ is a conformal metric on X, which can be thought of as a "circular average" of $|\beta|$,

$$|\beta^{1,1}(v)| = \frac{1}{2\pi} \int_0^{2\pi} |\beta(R_\theta v, R_\theta v)| d\theta,$$

where $v \in T_x X$ and $R_\theta \in \mathrm{Aut}(T_x X)$ is the rotation by angle θ defined by the conformal structure of X. The area of X with respect to $|\beta^{1,1}|$, when it is finite, defines a natural L^1 norm

$$\|\beta\|_{L^1(X)} = \int_X |\beta^{1,1}|,$$

which we will abbreviate to $\|\beta\|_1$ if the domain X is fixed. We also call $\|\beta\|_1$ the *energy* of β.

We write $S^{2,0}(X)$ for the space of *measurable quadratic differentials* on X, which are those quadratic forms $\phi \in S(X)$ such that $\phi = \phi^{2,0}$. Within $S^{2,0}(X)$, there is the space $Q(X)$ of *holomorphic quadratic differentials*, i.e. holomorphic sections of K_X^2. By the Riemann-Roch theorem, if X is compact and has genus g, $\dim_{\mathbb{C}} Q(X) = 3g - 3$.

Fixing a conformal metric σ also allows us to define the unit tangent bundle

$$UTX = \{(x, v) \in TX \mid \|v\|_\sigma = 1\},$$

in which case the interpretation of $\beta^{1,1}$ as the circular average of β yields another expression for $\|\beta\|_{L^1(X)}$:

$$\|\beta\|_{L^1(X)} = \frac{1}{2\pi} \int_X \int_{UT_xX} |\beta(v)| d\theta(v) d\sigma(x)$$

The *Hopf differential* $\Phi(\beta)$ of $\beta \in S(X)$ is the $(2,0)$ part of its decomposition,

$$\Phi(\beta) = \beta^{2,0},$$

which (along with $\Phi(\overline{\beta})$) measures the failure of β to be compatible with the conformal structure of X. For example, there is a function $b : X \to \mathbb{C}$ such that $\beta = b\sigma$ if and only if $\Phi(\beta) = \Phi(\overline{\beta}) = 0$.

Let $f : X \to (M, \rho)$ be a smooth map from X to a Riemannian manifold (M, ρ). Then the energy $\mathscr{E}(f)$ and Hopf differential $\Phi(f)$ of f are defined to be those of the pullback metric $f^*(\rho)$:

$$\begin{aligned} \mathscr{E}(f) &= \|f^*(\rho)\|_1 = \int_X |f^*(\rho)^{1,1}| \\ \Phi(f) &= [f^*(\rho)]^{2,0} \end{aligned} \tag{1}$$

Since $f^*(\rho)$ is a real quadratic form, at points where Df is nondegenerate f is conformal if and only if $\Phi(f) = 0$. Thus $\mathscr{E}(f)$ is a measure of the average stretching of the map f, while $\Phi(f)$ records its anisotropy.

4. Measured foliations and the antipodal map

In this section we briefly recall the identifications between the spaces of measured foliations, measured geodesic laminations, and holomorphic quadratic differentials on a compact Riemann surface. We then use these identifications to define the antipodal map on the space of measured laminations.

A measured foliation on S is a singular foliation $\mathscr{F}$ (i.e. one with isolated k-pronged singularities) and an assignment of a Borel measure to each transverse arc in a manner compatible with transversality-preserving isotopy. A detailed discussion of measured foliations can be found in [FLP]. The notation $\mathscr{MF}(S)$ is used for the quotient of the set of measured foliations by the equivalence relation generated by isotopy and Whitehead moves (e.g., collapsing leaves connecting singularities).

The typical example of a measured foliation comes from a holomorphic quadratic differential $\phi \in Q(X)$. The measured foliation $\mathscr{F}(\phi)$ determined by ϕ is the pullback of the horizontal line foliation of $\mathbb{C}$ under integration of the locally defined holomorphic 1-form $\sqrt{\phi}$. Equivalently, a vector $v \in T_x X$ is tangent to $\mathscr{F}(\phi)$ if and only if $\phi(v) > 0$. The measure on transversals is obtained by integrating the length element $|\operatorname{Im} \sqrt{\phi}|$.

The foliation $\mathscr{F}(\phi)$ is called the *horizontal foliation* of ϕ. Since $\sqrt{-\phi} = i\sqrt{\phi}$, $\mathscr{F}(\phi)$ and $\mathscr{F}(-\phi)$ are orthogonal, and $\mathscr{F}(-\phi)$ is called the *vertical foliation* of ϕ.

Hubbard and Masur proved that measured foliations and quadratic differentials are essentially equivalent notions:

Theorem 4.1 (Hubbard and Masur, [HM]). *For each $\nu \in \mathscr{MF}(S)$ of measured foliations and $X \in \mathscr{T}(S)$ there is a unique holomorphic quadratic differential $\phi_X(\nu) \in Q(X)$ such that*

$$\mathscr{F}(\phi_X(\nu)) = \nu.$$

Furthermore, the map $\phi_X : \mathscr{MF}(S) \to Q(X)$ is a homeomorphism.

Note that the transverse measure of $\phi \in Q(X)$ is defined using $\sqrt{\phi}$, and so for $c > 0$,

$$\mathscr{F}(c\phi) = c^{\frac{1}{2}} \mathscr{F}(\phi).$$

As a result, the Hubbard-Masur map ϕ_X has the following homogeneity property:

$$\phi_X(C\nu) = C^2 \phi_X(\nu) \text{ for all } C > 0 \tag{2}$$

One can also view measured foliations as diffuse versions of measured laminations, in that every measured foliation is associated to a unique measured lamination with the same intersection properties (for details see [Lev]). This induces a homeomorphism between $\mathscr{MF}(S)$ and $\mathscr{ML}(S)$. Using this homeomorphism implicitly, we can consider the Hubbard-Masur map ϕ_X to have domain $\mathscr{ML}(S)$, and we write Λ for its inverse,

$$\Lambda : Q(X) \to \mathscr{ML}(S).$$

We can also use Λ to transport the linear involution $\phi \mapsto (-\phi)$ of $Q(X)$ to an involutive homeomorphism $i_X : \mathscr{ML}(S) \to \mathscr{ML}(S)$, i.e.

$$i_X(\lambda) = \Lambda(-\phi_X(\lambda)).$$

Since $\mathscr{F}(\phi)$ and $\mathscr{F}(-\phi)$ are orthogonal foliations, we say that $\lambda, \mu \in \mathscr{ML}(S)$ are *orthogonal with respect to X* if $i_X(\lambda) = \mu$.

The resulting homeomorphism depends on $X \in \mathscr{T}(S)$ in an essential way, just as the orthogonality of foliations or tangent vectors depends on the choice of a conformal structure.

Since Λ is homogeneous, it also induces a homeomorphism between projective spaces:

$$\Lambda : \mathbb{P}^+Q(X) = (Q(X) - \{0\})/\mathbb{R}^+ \to \mathbb{P}\mathscr{ML}(S) = (\mathscr{ML}(S) - \{0\})/\mathbb{R}^+.$$

Thus we obtain an involution $i_X : \mathbb{P}\mathscr{ML}(S) \to \mathbb{P}\mathscr{ML}(S)$ that is topologically conjugate to the antipodal map $(-1) : \mathbb{P}^+Q(X) \simeq S^{2n-1} \to S^{2n-1}$. We call i_X the *antipodal involution with respect to* X.

5. $\mathbb{R}$-trees from measured laminations and foliations

An $\mathbb{R}$-tree (or *real tree*) is a complete geodesic metric space in which there is a unique embedded path joining every pair of points, and each such path is isometric to an interval in $\mathbb{R}$. Such $\mathbb{R}$-trees arise naturally in the context of measured foliations on surfaces, as we now describe (for a detailed treatment, see [Kap]).

Let $\mathscr{F} \in \mathscr{MF}(S)$ be a measured foliation on S, and lift $\mathscr{F}$ to a measured foliation $\tilde{\mathscr{F}}$ of the universal cover $\tilde{S} \simeq \mathbb{H}^2$. Define a pseudometric $d_{\mathscr{F}}$ on $\tilde{S}$,

$$d_{\mathscr{F}}(x,y) = \inf \{ i(\tilde{\mathscr{F}}, \gamma) \mid \gamma : [0,1] \to \tilde{S},\ \gamma(0) = x,\ \gamma(1) = y \} \tag{3}$$

where $i(\tilde{\mathscr{F}}, \gamma)$ is the intersection number of a transverse arc γ with the measured foliation $\tilde{\mathscr{F}}$.

The quotient metric space $T_{\mathscr{F}} = \tilde{S}/(x \sim y \text{ if } d_{\mathscr{F}}(x,y) = 0)$ is an $\mathbb{R}$-tree whose isometry type depends only on the measure equivalence class of $\mathscr{F}$. Alternately, $T_{\mathscr{F}}$ is the space of leaves of $\tilde{\mathscr{F}}$ with metric induced by the transverse measure, where we consider all of the leaves that emanate from a singular point of $\tilde{\mathscr{F}}$ to be a single point of $T_{\mathscr{F}}$. The action of $\pi_1(S)$ on $\tilde{S}$ by deck transformations descends to an action on $T_{\mathscr{F}}$ by isometries; in fact, by Skora's theorem, every such isometric action of $\pi_1(S)$ on an $\mathbb{R}$-tree satisfying a nondegeneracy condition (*small* and *minimal*) arises in this way (see [Sko], also [FW], [Ota]).

If $\mathscr{F}$ is a measured foliation associated to the measured lamination λ, then the resulting $\mathbb{R}$-tree T_λ with metric d_λ is called the *dual* $\mathbb{R}$*-tree* of λ. For example, when λ is supported on a family of simple closed curves, the $\mathbb{R}$-tree T_λ is actually a simplicial tree of infinite valence with one vertex for each lift of a complementary region of λ to $\tilde{S}$, and where an edge connecting two adjacent complementary regions has length equal to the weight of the geodesic that separates them.

A slight generalization of this construction arises naturally in the context of grafting. The grafting locus $A = \kappa^{-1}(\lambda) \subset \mathrm{gr}_\lambda Y$ has a natural foliation $\mathscr{F}_A$ by Euclidean geodesics that map isometrically onto λ, with a transverse measure induced by the Euclidean metric in the orthogonal direction. The associated pseudometric on $\widetilde{\mathrm{gr}_\lambda Y}$,

$$d_{\mathscr{F}_A}(x,y) = \inf \{ i(\tilde{\mathscr{F}}_A, \gamma) \mid \gamma : [0,1] \to \widetilde{\mathrm{gr}_\lambda Y},\ \gamma(0) = x,\ \gamma(1) = y \},$$

yields a quotient $\mathbb{R}$-tree isometric to T_λ and a map

$$\hat{\kappa} : \widetilde{\mathrm{gr}_\lambda Y} \to T_\lambda,$$

which we call the *co-collapsing map.* While the collapsing map $\kappa : \mathrm{gr}_\lambda Y \to Y$ compresses the entire grafted part back to its geodesic representative, the co-collapsing map collapses each connected component of $(\widetilde{\mathrm{gr}_\lambda Y} - \tilde{A})$, i.e the complement of the grafted part of $\widetilde{\mathrm{gr}_\lambda Y}$, and each leaf of $\mathscr{F}_A$ to a single point.

6. Harmonic maps to surfaces and $\mathbb{R}$-trees

We now consider the harmonic maps variational problem, i.e. finding critical points of the energy functional $\mathscr{E}(f)$ for maps $f : X \to (M, \rho)$ from a Riemann surface X to a Riemannian manifold M. In §3, we defined the energy for smooth maps, but the natural setting in which to work with the energy functional is the Sobolev space $W^{1,2}(X, M)$ of maps with L^2 distributional derivatives (see [KS1] for details). The maps we consider are Lipschitz, hence differentiable almost everywhere, so there is no ambiguity in our definitions.

A critical point of the energy functional $\mathscr{E} : W^{1,2}(X, M) \to \mathbb{R}^{\geq 0}$ is a *harmonic map*; it is easy to show that the Hopf differential

$$\Phi(f) = f^*(\rho)^{2,0}$$

is holomorphic ($\Phi(f) \in Q(X)$) if f is harmonic. In particular, the maximal and minimal stretch directions of a harmonic map are realized as a pair of orthogonal foliations by straight lines in the singular Euclidean metric $|\Phi(f)|$.

For any pair of compact hyperbolic surfaces X, Y and nontrivial homotopy class of maps $[f] : X \to Y$, there is a unique harmonic map $h : X \to Y$ that is smooth and homotopic to f; furthermore, h minimizes energy among such maps [Har]. In particular, for each $X, Y \in \mathscr{T}(S)$ there is a unique harmonic map $h : X \to Y$ that is compatible with the markings; define

$$\mathscr{E}(X, Y) = \mathscr{E}(h : X \to Y).$$

For the proof of Theorem 1.1 we will also need to consider harmonic maps from Riemann surfaces to $\mathbb{R}$-trees. The main references for this theory are the papers of Wolf ([W2], [W3], [W4]); a much more general theory of harmonic maps to metric spaces is discussed by Korevaar and Schoen in [KS1] and [KS2]. The $\mathbb{R}$-trees and harmonic maps we consider will arise from limits of maps to degenerating hyperbolic surfaces; with the appropriate notion of convergence, Wolf has shown that an $\mathbb{R}$-tree limit can always be extracted:

Theorem 6.1 (Wolf [W2]). *Let $Y_i \in \mathscr{T}(S)$ and $\mu \in \mathscr{ML}(S)$ be such that $Y_i \to [\mu] \in \mathbb{P}\mathscr{ML}(S)$ in the Thurston compactification. Then after rescaling the hyperbolic metrics ρ_i on Y_i appropriately, the sequence of universal covering metric spaces $(\tilde{Y}_i, \tilde{\rho}_i)$ converges in the equivariant Gromov-Hausdorff sense to the $\mathbb{R}$-tree T_μ.*

The equivariant Gromov-Hausdorff topology is a natural setting in which to consider convergence of metric spaces equipped with isometric group actions. The application of this topology to the Thurston compactification is due to Paulin [Pau]; for other perspectives on the connection between $\mathbb{R}$-trees and Teichmüller theory, see [Bes], [MS], [CS].

The theory of harmonic maps is well-adapted to this generalization from smooth surfaces to metric spaces like $\mathbb{R}$-trees; for example, the convergence statement of Theorem 6.1 can be extended to a family of harmonic maps from a fixed Riemann surface X:

Theorem 6.2 (Wolf [W2]). *For $X \in \mathscr{T}(S)$ and $\lambda \in \mathscr{ML}(S)$, let $\pi_\lambda : \tilde{X} \to T_\lambda$ denote the projection onto the leaves of $\mathscr{F}(\phi_X(\lambda))$, where $\phi_X(\lambda)$ is the Hubbard-Masur differential for λ. Then:*

1. *π_λ is harmonic, meaning that it pulls back germs of convex functions on T_λ to germs of subharmonic functions on $\tilde{X}$.*
2. *If $Y_i \in \mathscr{T}(S)$ is a sequence such that $Y_i \to [\lambda] \in \mathbb{P}\mathscr{ML}(S)$, then the lifts $\tilde{h}_i : \tilde{X} \to \tilde{Y}_i$ of the harmonic maps converge in the equivariant Gromov-Hausdorff sense to $\pi_X : \tilde{X} \to T_\lambda$.*

Much like the case of maps between Riemann surfaces, it is most natural to work with the energy functional on a Sobolev space $W^{1,2}(\tilde{X}, T_\lambda)$ of equivariant maps with L^2 distributional derivatives, and again we defer to [KS1] for details, since the maps we consider are Lipschitz.

By definition, the metric d_λ on T_λ is isometric to the standard Euclidean metric dx^2 on $\mathbb{R}$ along each geodesic segment. Thus if $f : \tilde{X} \to T_\lambda$ is an equivariant Lipschitz map that sends a neighborhood of almost every point in $\tilde{X}$ into a geodesic segment of T_λ, then the pullback metric $f^*(dx^2)$ is a well-defined (possibly degenerate) measurable quadratic form on $T\tilde{X}$ which is invariant under the action of $\pi_1(X)$. This allows us to define the energy $\mathscr{E}(f)$ and Hopf differential $\Phi(f)$ of such an equivariant map as the L^1 norm and $(2,0)$ part of the induced quadratic form on TX.

For later use we record the following calculations relating the Hopf differential and energy of the projection π_λ and the Hubbard-Masur differential $\phi_X(\lambda)$; details can be found in [W2].

Lemma 6.3. *The Hopf differential of $\pi_\lambda : \tilde{X} \to T_\lambda$ is*

$$\Phi(\pi_\lambda) = -\phi_X(\frac{1}{2}\lambda) = -\frac{1}{4}\phi_X(\lambda),$$

and the energy of π_λ is given by

$$\mathscr{E}(\pi_\lambda) = \frac{1}{2}E(\lambda, X) = \frac{1}{2}\|\phi_X(\lambda)\|_1,$$

where $E(\lambda, X)$ is the extremal length of λ on X.

7. Energy and grafting

In this section we recall Tanigawa's computation of the energy of the collapsing map and discuss its consequences for grafting and pruning. Tanigawa used this energy computation to show that for each $\lambda \in \mathscr{ML}(S)$, grafting $\mathrm{gr}_\lambda : \mathscr{T}(S) \to \mathscr{T}(S)$ is a proper map of Teichmüller space.

Lemma 7.1 (Tanigawa [Tan])**.** *Let $X = \mathrm{gr}_\lambda Y$, where $X, Y \in \mathscr{T}(S)$ and $\lambda \in \mathscr{ML}(S)$, and let h denote the harmonic map $h : X \to Y$ compatible with the markings (and thus homotopic to $\kappa : X \to Y$). Then*

$$\frac{1}{2}\ell(\lambda, Y) \leq \frac{1}{2}\frac{\ell(\lambda, Y)^2}{E(\lambda, X)} \leq \mathscr{E}(h) \leq \frac{1}{2}\ell(\lambda, Y) + 2\pi|\chi(S)| = \mathscr{E}(\kappa),$$

where $\ell(\lambda, Y)$ is the hyperbolic length of λ on Y, and $E(\lambda, X)$ is the extremal length of λ on X.

The middle part of Tanigawa's inequality, i.e.

$$\frac{1}{2}\frac{\ell(\lambda, Y)^2}{E(\lambda, X)} \leq \mathscr{E}(h)$$

is due to Minsky, and holds for any harmonic map between finite-area Riemann surfaces of the same type and any measured lamination λ [Min].

We will also need to know the energy of the co-collapsing map, which is easily computed using the same method as in [Tan]:

Lemma 7.2. *Let $X = \mathrm{gr}_\lambda Y$, where $X, Y \in \mathscr{T}(S)$ and $\lambda \in \mathscr{ML}(S)$. Then the energy $\mathscr{E}(\hat{\kappa})$ of the co-collapsing map $\hat{\kappa} : \tilde{X} \to T_\lambda$ is given by*

$$\mathscr{E}(\hat{\kappa}) = \frac{1}{2}\ell(\lambda, Y).$$

Proof. On the grafting locus, both the collapsing and co-collapsing maps are locally modeled on the orthogonal projection of a Euclidean plane onto a geodesic. Thus the conformal parts of the pullback metrics associated to the collapsing and co-collapsing maps are identical on the grafting locus, i.e.

$$[\kappa^*(\rho)]^{1,1} = [\hat{\kappa}^*(\rho)]^{1,1}.$$

On the remainder of the surface (the hyperbolic part), the pullback metric via the co-collapsing map is zero, while the collapsing map is an isometry here. The energy difference is therefore the hyperbolic area of Y,

$$\mathscr{E}(\kappa) = \mathscr{E}(\hat{\kappa}) + 2\pi|\chi(S)|,$$

and the lemma follows. $\square$

It is an immediate consequence of Lemma 7.1 that the the collapsing map has energy very close to that of the harmonic map:

Corollary 7.3. *Let $h : \mathrm{gr}_\lambda Y \to Y$ be the harmonic map homotopic to the collapsing map κ. Then*

$$\mathscr{E}(h) < \mathscr{E}(\kappa) \leq \mathscr{E}(h) + 2\pi|\chi(S)|.$$

Note that the constant on the right hand side is independent of $Y \in \mathscr{T}(S)$ and $\lambda \in \mathscr{ML}(S)$.

For our purposes, another important consequences of Lemma 7.1 is a relationship between the hyperbolic length of the grafting lamination on Y and its extremal length on $\mathrm{gr}_\lambda Y$:

Lemma 7.4. *Let $X = \mathrm{gr}_\lambda Y$, where $Y \in \mathscr{T}(S)$. Then we have*

$$\ell(\lambda, Y) = E(\lambda, X) + O(1)$$

where the implicit constant depends only on $|\chi(S)|$.

Proof. The lower bound

$$\ell(\lambda, Y) \geq E(\lambda, X)$$

is immediate from the left-hand side of the inequality of Lemma 7.1, and we also have

$$\frac{1}{2}\frac{\ell(\lambda, Y)^2}{E(\lambda, X)} \leq \frac{1}{2}\ell(\lambda, Y) + 2\pi|\chi(S)|$$

and therefore,

$$\frac{\ell(\lambda, Y)^2}{\frac{1}{2}\ell(\lambda, Y) + O(1)} \leq E(\lambda, X).$$

Solving for $\ell(\lambda, Y)$ yields

$$\begin{aligned} \ell(\lambda, Y) &\leq \frac{1}{2}E(\lambda, X) + \frac{1}{2}(E(\lambda, X))^{1/2}(E(\lambda, X) + O(1))^{1/2} \\ &= E(\lambda, X) + O(1) \end{aligned}$$

∎

Using Lemma 7.2 and this comparison between hyperbolic and extremal length, we find that the energy of the co-collapsing map is close to that of the harmonic projection (cf. Corollary 7.3):

Corollary 7.5. *Let π_λ be the harmonic projection from $\tilde{X} = \widetilde{\mathrm{gr}_\lambda Y}$ to the $\mathbb{R}$-tree T_λ, and $\hat{\kappa}$ the associated co-collapsing map. Then*

$$\mathscr{E}(\pi_\lambda) < \mathscr{E}(\hat{\kappa}) \leq \mathscr{E}(\pi_\lambda) + 2\pi|\chi(S)|.$$

Finally, as a complement to Tanigawa's result on grafting, we can use these energy computations to show that the pruning map with basepoint X is proper:

Lemma 7.6. *For each $X \in \mathscr{T}(S)$, the pruning map with basepoint X, i.e.*

$$\mathrm{pr}_{(\cdot)} X : \mathscr{ML}(S) \to \mathscr{T}(S)$$

is proper.

Proof. Suppose on the contrary that $\lambda_i \to \infty$ but the sequence $Y_i = \mathrm{pr}_{\lambda_i} X$ remains in a compact subset of Teichmüller space. Then $\ell(\lambda_i, Y_i) \to \infty$ and by Lemma 7.1,

$$\mathscr{E}(h_i) \geq \frac{1}{2}\ell(\lambda_i, Y_i) \to \infty,$$

where $h_i : X \to Y_i$ is the harmonic map compatible with the markings.

On the other hand, a result of Wolf (see [W1]) states that for any fixed $X \in \mathscr{T}(S)$, the energy $\mathscr{E}(X, \cdot)$ is a proper function on $\mathscr{T}(S)$. Since $\mathscr{E}(h_i) \to \infty$, we conclude $Y_i \to \infty$, which is a contradiction. ∎

8. Hopf differentials and grafting

We will need to consider not only the energy but also the Hopf differentials $\Phi(\kappa)$ and $\Phi(\hat{\kappa})$ associated to the collapsing and co-collapsing maps of a grafted surface. In particular, for the proof of Theorem 1.1, we will use a relationship between these differentials and the grafting lamination. In this section we establish such a relationship, after addressing some technical issues that arise because the quadratic differentials under consideration are not holomorphic.

For holomorphic quadratic differentials $\phi, \psi \in Q(X)$, the intersection number of their measured foliations can be expressed in terms of the differentials (see [Gar]); define

$$\omega(\phi, \psi) = \int_X |\operatorname{Im}\left(\sqrt{\phi}\sqrt{\psi}\right)|.$$

Then

$$i(\mathscr{F}(\phi), \mathscr{F}(\psi)) = \omega(\phi, \psi).$$

However, the quantity $\omega(\alpha, \beta)$ makes sense for L^1 quadratic differentials α and β, holomorphic or not.

While a measurable differential α does not define a measured foliation, it does have a *horizontal line field* $\mathscr{L}(\alpha)$ consisting of directions $v \in TX$ such that $\alpha(v) > 0$. Then $\omega(\alpha, \beta)$ measures the average transversality (sine of twice the angle) between the line fields $\mathscr{L}(\alpha)$ and $\mathscr{L}(\beta)$, averaged with respect to the measure $|\alpha|^{1/2}|\beta|^{1/2}$.

Now consider the collapsing map $\kappa : X \to \operatorname{pr}_\lambda X$, and for simplicity let us first suppose λ is supported on a single simple closed hyperbolic geodesic $\gamma \subset \operatorname{pr}_\lambda X$, i.e. $\lambda = t\gamma$. Then the grafting locus $A \subset X$ is the Euclidean cylinder $\gamma \times [0, t]$, and the collapsing map is the projection onto the geodesic γ. Just as the Hopf differential of the orthogonal projection of $\mathbb{C}$ onto $\mathbb{R}$ is

$$\Phi(z \mapsto \operatorname{Re}(z)) = [dx^2]^{2,0} = \frac{1}{4} dz^2,$$

the Hopf differential of κ on A is the pullback of $\frac{1}{4}dz^2$ via local Euclidean charts that take parallels of γ to horizontal lines. This differential is holomorphic on A, and corresponds to the measured foliation $\frac{1}{2}\lambda$. Thus the Euclidean metric on A, which is the restriction of the Thurston metric of X, is given by $|4\Phi(\kappa)|$.

On the complement of the grafting locus, the collapsing map is conformal and thus the Hopf differential is zero. Therefore $\Phi(\kappa)$ is a piecewise holomorphic differential on X whose horizontal line field is the natural foliation of the grafting locus by parallels of the grafting lamination, with half of the measure of λ. This analysis extends by continuity to the case of a general lamination $\lambda \in \mathscr{ML}(S)$.

It follows that the line field $\mathscr{L}(\Phi(\kappa))$ represents the measured lamination $\frac{1}{2}\lambda$, in that for all $\psi \in Q(X)$,

$$\omega(\Phi(\kappa), \psi) = \frac{1}{2} i(\lambda, \mathscr{F}(\psi)). \tag{4}$$

We therefore use the notation

$$\Phi_X(\lambda) = \Phi(\kappa : X \to \mathrm{pr}_\lambda X)$$

for the Hopf differential of the collapsing map, which is somewhat like $\phi_X(\frac{1}{2}\lambda)$ in that it is a quadratic differential whose foliation is a distinguished representative for the measured foliation class of $\frac{1}{2}\lambda$. The Hopf differential $\Phi_X(\lambda)$ is not holomorphic, however, though we will later see (§10) that is is nearly so.

For now, we will simply show that L^1 convergence of Hopf differentials $\Phi_X(\lambda)$ to a holomorphic limit implies convergence of the laminations λ:

Lemma 8.1. *Let $X \in \mathscr{T}(S)$ and $\lambda_i \in \mathscr{ML}(S)$. If*

$$[\Phi_X(\lambda_i)] \to [\psi], \quad \text{where } \psi \in Q(X)$$

then

$$[\lambda_i] \to [\mathscr{F}(\psi)] \in \mathbb{P}\mathscr{ML}(S).$$

Here $[\Phi_X(\lambda_i)]$ is the image of $\Phi_X(\lambda_i)$ in $\mathbb{P}S^{2,0}(X)$.

Proof. First, we can choose $c_i > 0$ such that

$$c_i^2 \Phi_X(\lambda_i) \to \psi.$$

It is well known that there are finitely many simple closed curves ν_k, $k = 1 \ldots N$, considered as measured laminations with unit weight, such that the map $I : \mathscr{ML}(S) \to \mathbb{R}^N$ defined by $I(\lambda) = \{i(\lambda, \nu_k)\}$ is a homeomorphism onto its image. Recall that $\phi_X(\nu_k) \in Q(X)$ is the unique holomorphic quadratic differential satisfying $\mathscr{F}(\phi_X(\nu_k)) = \nu_k$.

Since $\omega(\cdot, \nu_k) : S^{2,0}(X) \to \mathbb{R}$ is evidently a continuous map, we conclude from (4) and the hypothesis $c_i \Phi_X(\lambda_i) \to \psi$ that

$$\omega(c_i^2 \Phi_X(\lambda_i), \phi_X(\nu_k)) = \frac{c_i}{2} i(\lambda_i, \nu_k) \to \frac{1}{2} i(\mathscr{F}(\psi), \nu_k).$$

and therefore $c_i \lambda_i \to \frac{1}{2}\mathscr{F}(\psi)$. ∎

The connection between the Hopf differential $\Phi_X(\lambda)$ and the Thurston metric also allows us to compute its norm and relate it to the extremal length of λ:

Corollary 8.2. *The L^1 norm of $\Phi_X(\lambda)$ is given by*

$$\|\Phi_X(\lambda)\|_1 = \frac{1}{4}\ell(\lambda, \mathrm{pr}_\lambda X) = \frac{1}{4}E(\lambda, X) + O(1).$$

Proof. We have seen that $|4\Phi_X(\lambda)|$ induces the Thurston metric on the grafted part $A \subset X$ and is zero elsewhere. The area of A with respect to the Thurston metric is $\ell(\lambda, \mathrm{pr}_\lambda X)$, and therefore,

$$\|\Phi_X(\lambda)\|_1 = \frac{1}{4}\ell(\lambda, \mathrm{pr}_\lambda X).$$

On the other hand, it follows from Lemma 7.4 that

$$\ell(\lambda, \mathrm{pr}_\lambda X) = E(\lambda, X) + O(1).$$

□

We can apply the same analysis to the Hopf differential of the co-collapsing map $\hat{\kappa} : \widetilde{\mathrm{Gr}_\lambda Y} \to T_\lambda$. Though the co-collapsing map is defined on the universal cover of the grafted surface, its Hopf differential is invariant under the action of $\pi_1(S)$ and therefore descends to a measurable quadratic differential $\Phi(\hat{\kappa}) \in S^{2,0}(X)$.

The co-collapsing map $\hat{\kappa}$ is piecewise constant on the complement of the grafting locus, so (like $\Phi(\kappa)$) its Hopf differential is identically zero there. Within the grafting locus it is modeled on the orthogonal projection of $\mathbb{C}$ onto $i\mathbb{R}$ (where the leaves of $\mathscr{F}_A$ correspond to horizontal lines in $\mathbb{C}$). Since

$$\Phi(z \mapsto \mathrm{Im}(z)) = [dy^2]^{2,0} = -\frac{1}{4}dz^2,$$

we conclude that the Hopf differentials $\Phi(\hat{\kappa})$ and $\Phi(\kappa)$ are inverses, i.e.

$$\Phi(\hat{\kappa}) = -\Phi(\kappa) = -\Phi_X(\lambda) \tag{5}$$

Remark. The relationship between κ and $\hat{\kappa}$ and their Hopf differentials is reminiscent of the "minimal suspension" technique introduced by Wolf; for details, see [W5].

9. Convergence to the harmonic map

We will use the fact that the collapsing map associated to a grafted surface is nearly harmonic to extract a genuinely harmonic limit map with values in an $\mathbb{R}$-tree. In this section we formalize the conditions on a sequence of maps that allows us to use this limit construction and prove the main technical result about convergence.

Let $X, Y_i \in \mathscr{T}(S)$ and suppose $Y_i \to \infty$; let ρ_i denote a hyperbolic metric on Y_i, and $h_i : X \to Y_i$ the harmonic map (with respect to ρ_i) compatible with the markings.

We say that a sequence of maps $f_i \in W^{1,2}(X, Y_i)$ compatible with the markings of X and Y_i is a *minimizing sequence* if

$$\lim_{i\to\infty} \frac{\mathscr{E}(f_i)}{\mathscr{E}(h_i)} = 1. \tag{6}$$

Since the harmonic map h_i is the unique energy minimizer in its homotopy class, a minimizing sequence asymptotically minimizes energy. In this section we will show that all minimizing sequences have the same asymptotic behavior, in that their Hopf differentials have the same projective limit.

Note that the condition that f_i be a minimizing sequence implies only that the energy differences $\mathscr{E}(f_i) - \mathscr{E}(h_i)$ are $o(\mathscr{E}(h_i))$, while in the proof of Theorem 1.1 we consider a sequence of collapsing maps for which these energy differences are $O(\mathscr{E}(h_i)^{\frac{1}{2}})$ (while $\mathscr{E}(h_i) \to \infty$), a stronger condition. We prove the next result for general minimizing sequences, however, since a specific bound on the energy difference does not substantially simplify its proof.

Theorem 9.1. *Let X and Y_i be as above, and suppose $\lim_{i\to\infty} Y_i = [\mu] \in \mathbb{P}\mathscr{ML}(S)$ in the Thurston compactification.*

Then for any minimizing sequence $f_i : X \to Y_i$, the measurable quadratic differentials $[f_i^*(\rho_i)]^{2,0}$ converge projectively in the L^1 sense to a holomorphic quadratic differential $\Phi \in Q(X)$ such that

$$[\mathscr{F}(-\Phi)] = [\mu],$$

i.e. there are constants $c_i > 0$ such that

$$\lim_{i\to\infty} c_i [f_i^*(\rho_i)]^{2,0} = \Phi.$$

Note. The vertical foliation $\mathscr{F}(-\Phi)$ appears in Theorem 9.1 because the Thurston limit is a lamination whose intersection number provides an estimate of hyperbolic length; directions orthogonal to $\mathscr{F}(-\Phi)$ (that is, tangent to $\mathscr{F}(\Phi)$) are maximally stretched by a map with Hopf differential Φ, so the intersection number with $\mathscr{F}(-\Phi)$ provides such a length estimate.

Before giving the proof of Theorem 9.1, we recall a theorem of Wolf upon which it is based.

Theorem 9.2 (Wolf, [W1])**.** *Let $X, Y_i \in \mathscr{T}(S)$, and let Ψ_i denote the Hopf differential of the unique harmonic map $h_i : X \to Y_i$ respecting markings, where Y_i is given the hyperbolic metric ρ_i. Then*

$$\lim_{i\to\infty} Y_i = [\mu] \in \mathbb{P}\mathscr{ML}(S)$$

if and only if

$$\lim_{i\to\infty} [\mathscr{F}(-\Psi_i)] = [\mu].$$

In other words, if one compactifies Teichmüller space according to the limiting behavior of the Hopf differential of the harmonic map from a fixed conformal structure X, then the vertical foliation map $\mathscr{F} \circ (-1) : \mathbb{P}^+Q(X) \to \mathbb{P}\mathscr{MF}(S) \simeq \mathbb{P}\mathscr{ML}(S)$ identifies this compactification with the Thurston compactification.

Theorem 9.2 is actually a consequence of the convergence of harmonic maps h_i to the harmonic projection π_μ to an $\mathbb{R}$-tree (combining Theorem

6.1 and Theorem 6.2), though in [W1] Wolf provides an elementary and streamlined proof of this result.

We will compare the Hopf differentials of a minimizing sequence to those of a harmonic map by means of the pullback metrics. The essential point is that the energy functional for maps to a nonpositively curved (NPC) space is convex along geodesic homotopies. Korevaar and Schoen use this to show that maps with nearly minimal energy have nearly the same pullback metric:

Theorem (Korevaar-Schoen, [KS1, §2.6])**.** *Let X be a compact Riemann surface and Z an NPC metric space equipped with an isometric action of $\pi_1(X)$. Let $f \in W^{1,2}(\tilde{X}, Z)$ be an equivariant map and $h \in W^{1,2}(\tilde{X}, Z)$ an equivariant energy-minimizing map. Then*

$$\frac{1}{2\pi}\int_{UTX} |\pi_f(v,v) - \pi_h(v,v)| \leq (2\mathscr{E}(f) - 2\mathscr{E}(h))^{\frac{1}{2}} \left(\mathscr{E}(f)^{\frac{1}{2}} + \mathscr{E}(h)^{\frac{1}{2}}\right) \tag{7}$$

Here π_f is the quadratic form that plays the role of the metric pulled back by f in the Korevaar-Schoen theory of Sobolev maps to NPC metric spaces (see [KS1, §2]). Note that we have specialized this theorem to the case of a compact surface domain, whereas the actual estimate in [KS1] applies to any compact Riemannian manifold.

Applying this theorem to the case where Z the universal cover of a compact hyperbolic surface or an $\mathbb{R}$-tree, in which case $\pi_f(v,v) = f^*(\rho)(v)$ is the pullback metric considered above, we obtain the following estimates:

Corollary 9.3. *Let $f \in W^{1,2}(X,Y)$ where $X, Y \in \mathscr{T}(S)$ and Y is given the hyperbolic metric ρ. Let h be the harmonic map homotopic to f. Then*

$$\|f^*(\rho) - h^*(\rho)\|_1 \leq \sqrt{2}\,(\mathscr{E}(f) - \mathscr{E}(h))^{\frac{1}{2}} \left(\mathscr{E}(f)^{\frac{1}{2}} + \mathscr{E}(h)^{\frac{1}{2}}\right)$$

and in particular

$$\|\Phi(f) - \Phi(h)\|_1 \leq \sqrt{2}\,(\mathscr{E}(f) - \mathscr{E}(h))^{\frac{1}{2}} \left(\mathscr{E}(f)^{\frac{1}{2}} + \mathscr{E}(h)^{\frac{1}{2}}\right).$$

Corollary 9.4. *Let $f \in W^{1,2}(\tilde{X}, T_\lambda)$ be a π_1-equivariant map, where $X \in \mathscr{T}(S)$ and $\lambda \in \mathscr{ML}(S)$. Then*

$$\|\Phi(f) + \frac{1}{4}\phi_X(\lambda)\|_1 \leq \sqrt{2}\,(\mathscr{E}(f) - \mathscr{E}(\pi_\lambda))^{\frac{1}{2}} \left(\mathscr{E}(f)^{\frac{1}{2}} + \mathscr{E}(\pi_\lambda)^{\frac{1}{2}}\right)$$

Proof of Theorem 9.1. Clearly the sequence of harmonic maps $h_i : X \to Y_i$ is a minimizing sequence. Applying Theorem 9.2 to the sequence Y_i we find that the Hopf differentials converge projectively:

$$\lim_{i\to\infty} [\Phi(h_i)] = [\Phi_\infty], \text{ where } \mathscr{F}(-[\Phi_\infty]) = [\mu].$$

To prove Theorem 9.1, we therefore need only show that the (measurable) Hopf differentials $\Phi(f_i)$ of *any* minimizing sequence have the same projective limit as the holomorphic Hopf differentials $\Psi_i = \Phi(h_i)$ of the harmonic maps.

Applying Corollary 9.3 to such a sequence, we find

$$\begin{aligned}
\frac{\|f_i^*(\rho_i) - h_i^*(\rho_i)\|_1}{\mathscr{E}(h_i)} &\leq \frac{\sqrt{2}\,(\mathscr{E}(f_i) - \mathscr{E}(h_i))^{\frac{1}{2}} \left(\mathscr{E}(f_i)^{\frac{1}{2}} + \mathscr{E}(h_i)^{\frac{1}{2}}\right)}{\mathscr{E}(h_i)} \\
&= \sqrt{2}\left(\frac{\mathscr{E}(f_i)}{\mathscr{E}(h_i)} - 1\right)^{\frac{1}{2}} \left(1 + \left(\frac{\mathscr{E}(f_i)}{\mathscr{E}(h_i)}\right)^{\frac{1}{2}}\right).
\end{aligned} \tag{8}$$

Since f_i is a minimizing sequence, the first factor on the right hand side of (8) is $o(1)$ and the second is bounded, thus

$$\lim_{i\to\infty} \frac{\|f_i^*(\rho_i) - h_i^*(\rho_i)\|_1}{\mathscr{E}(h_i)} = 0.$$

Finally, the Hopf differential is the $(2,0)$ part of the pullback metric, so we have

$$\lim_{i\to\infty}[\Phi(f_i)] = \lim_{i\to\infty}[\Psi_i] = [\Phi], \text{ where } [\mathscr{F}(-\Phi)] = [\mu].$$

$\square$

10. Proof of the main theorem

In this section we apply Theorem 9.1 to the collapsing maps $\kappa_i : X \to Y_i$ to prove Theorem 1.1.

Proof of Theorem 1.1. Fix $X \in \mathscr{T}(S)$ and let $\lambda_i \in \mathscr{ML}(S)$ be a divergent sequence of measured laminations. Let $Y_i = \mathrm{pr}_{\lambda_i} X$ so that $X = \mathrm{gr}_{\lambda_i} Y_i$. By Lemma 7.6, the surfaces $Y_i \to \infty$ and thus $\mathscr{E}(X, Y_i) \to \infty$. We need to show that if $Y_i \to [\mu] \in \mathbb{P}\mathscr{ML}(S)$ and $[\lambda_i] \to [\lambda] \in \mathbb{P}\mathscr{ML}(S)$ then $i_X([\lambda]) = [\mu]$, or equivalently, that $[\lambda]$ and $[\mu]$ are the horizontal and vertical measured laminations of a single holomorphic quadratic differential on X.

By Corollary 7.3,

$$\mathscr{E}(\kappa_i) - \mathscr{E}(X, Y_i) = O(1),$$

while $\mathscr{E}(X, Y_i) \to \infty$, thus κ_i is a minimizing sequence (cf. §9). Applying Theorem 9.1, we conclude that

$$[\Phi_X(\lambda_i)] \to [\Phi] \in \mathbb{P}^+Q(X), \text{ where } [\mathscr{F}(-\Phi)] = [\mu].$$

On the other hand, by Lemma 8.1, this implies that $[\mathscr{F}(\Phi)] = [\lambda]$, and $i_X([\lambda]) = [\mu]$. $\square$

While Theorem 1.1 is an asymptotic statement, the ideas used in the proof above also yield the following finite version of the comparison between the Hopf differential of the collapsing map and the holomorphic differential $\phi_X(\lambda)$ representing the grafting lamination:

Theorem 10.1. *Let $X \in \mathscr{T}(S)$ and $\lambda \in \mathscr{ML}(S)$. Then the Hopf differential $\Phi_X(\lambda)$ of the collapsing map $\kappa : X \to \mathrm{pr}_\lambda X$ and the Hubbard-Masur differential $\phi_X(\lambda)$ satisfy*

$$\|4\Phi_X(\lambda) - \phi_X(\lambda)\|_1 \leq C\left(1 + E(\lambda, X)^{\frac{1}{2}}\right)$$

where C is a constant depending only on $\chi(S)$.

Proof. Recall from (5) that the Hopf differential of the co-collapsing map $\hat{\kappa}$ is

$$\Phi(\hat{\kappa}) = -\Phi_X(\lambda),$$

while by Lemma 6.3, that of the harmonic projection π_λ is

$$\Phi(\pi_\lambda) = -\frac{1}{4}\phi_X(\lambda).$$

By Corollary 7.5, the energy of $\hat{\kappa}$ satisfies

$$\mathscr{E}(\hat{\kappa}) - \mathscr{E}(\pi_\lambda) \leq A := 2\pi|\chi(S)|.$$

Therefore Corollary 9.4 implies that

$$\begin{aligned}
\| -\Phi_X(\lambda) + \frac{1}{4}\phi_X(\lambda)\|_1 &\leq \sqrt{2}\left(\mathscr{E}(\hat{\kappa}) - \mathscr{E}(\pi_\lambda)\right)^{\frac{1}{2}}\left(\mathscr{E}(\pi_\lambda)^{\frac{1}{2}} + \mathscr{E}(\hat{\kappa})^{\frac{1}{2}}\right)\\
&\leq \sqrt{2}A^{\frac{1}{2}}\left(\left(\tfrac{1}{2}E(\lambda,X)\right)^{\frac{1}{2}} + \left(\tfrac{1}{2}E(\lambda,X) + A\right)^{\frac{1}{2}}\right)\\
&\leq \sqrt{2}A^{\frac{1}{2}}\left(2\left(\tfrac{1}{2}E(\lambda,X)\right)^{\frac{1}{2}} + A^{\frac{1}{2}}\right)\\
&= \sqrt{2}A + 2A^{\frac{1}{2}}E(\lambda,X)^{\frac{1}{2}}
\end{aligned}$$

The Theorem then follows by algebra. □

A. Appendix: Asymmetry of Teichmüller geodesics

Since the antipodal involution i_X is a homeomorphism of $\mathbb{P}\mathscr{ML}(S)$ to itself, it seems natural to look for an involutive homeomorphism of $\mathscr{T}(S)$ that has i_X as its boundary values. In fact, an obvious candidate is the Teichmüller geodesic involution $I_X : \mathscr{T}(S) \to \mathscr{T}(S)$ that is the push-forward of $(-1) : Q(X) \to Q(X)$ via the Teichmüller exponential map $\tau : Q(X) \xrightarrow{\sim} \mathscr{T}(S)$. However, we now sketch an example showing that I_X does not extend continuously to the Thurston compactification of $\mathscr{T}(S)$, leading us to view the pruning map based at X, $\lambda \mapsto \mathrm{pr}_\lambda X$, as a kind of substitute for the Teichmüller geodesic involution that *does* extend continuously to the antipodal map of $\mathbb{P}\mathscr{ML}$.

By a theorem of Masur, if ϕ is a Strebel differential on X whose trajectories represent homotopy classes $(\alpha_1, \dots, \alpha_n)$, then the Teichmüller ray determined by ϕ converges to the point $[\alpha_1 + \cdots + \alpha_n] \in \mathbb{P}\mathscr{ML}(S)$ in the Thurston compactification [Mas2]. Note that the limit point corresponds to a measured lamination in which each curve α_i has the same weight, independent of the relative sizes of the cylinders on X determined by ϕ. This happens because the Thurston boundary reflects the geometry of hyperbolic geodesics on the surface, and hyperbolic length is approximated by the reciprocal of the logarithm of a cylinder's height, as in Figure 2.

Now suppose ϕ and ψ are holomorphic quadratic differentials on a Riemann surface X such that each of $\pm\phi, \pm\psi$ is Strebel, where the trajectories of ϕ and ψ represent different sets of homotopy classes, while those of $-\phi$

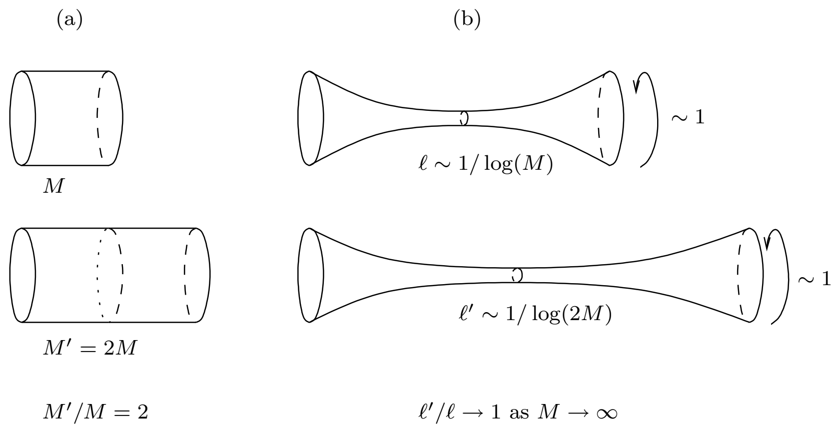

FIGURE 2. (a) Euclidean cylinders with moduli M and $2M$ correspond to (b) hyperbolic cylinders whose core geodesics have approximately the same length. This phenomenon leads to Teichmüller rays for distinct Strebel differentials that converge to the same point in the Thurston boundary of Teichmüller space [Mas2].

and $-\psi$ represent the same homotopy classes. Then by Masur's theorem, the Teichmüller geodesics corresponding to ϕ and ψ converge to the same point on $\mathbb{P}\mathscr{ML}(S)$ in the negative direction, while in the positive direction they converge to distinct points. If I_X were to extend to a continuous map of the Thurston boundary, then any pair of Teichmüller geodesics that are asymptotic in one direction would necessarily be asymptotic in both directions, thus no such extension exists.

One can explicitly construct such (X, ϕ, ψ) as follows: Let X_0 denote the square torus $\mathbb{C}/(2\mathbb{Z} \oplus 2i\mathbb{Z})$, and let $\psi_0 = \psi_0(z)dz^2$ be a meromorphic quadratic differential on X_0 with simple zeros at $z = \pm\frac{1+\epsilon}{2}$ and simple poles at $z = \pm\frac{1}{2}$ and such that $\psi_0(x) \in \mathbb{R}$ for $x \in \mathbb{R}$. Such a differential exists by the Abel-Jacobi theorem, and in fact is given by

$$\psi_0(z)dz^2 = \frac{\wp(z) - c_0}{\wp(z) - c_1}$$

for suitable constants c_i, where $\wp(z)$ is the Weierstrass function for X_0. Let $\phi_0 = dz^2$, a holomorphic quadratic differential on X_0. The trajectory structures of ϕ_0 and ψ_0 are displayed in Figure 3.

Let X be the surface of genus 2 obtained as a 2-fold cover of X_0 branched over $\pm\frac{1}{2}$; then ψ_0 and ϕ_0 determine holomorphic quadratic differentials ψ and ϕ on X, where the lift of ψ_0 is holomorphic because the simple poles at $\pm\frac{1}{2}$ are branch points of the covering map $X \to X_0$.

Both ϕ and ψ have closed vertical and horizontal trajectories as in the construction above (see Figure 4). Specifically, let γ and η denote the free

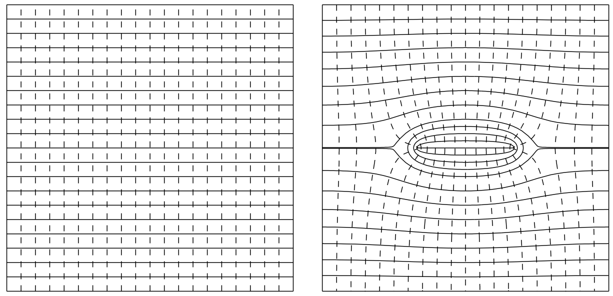

FIGURE 3. Horizontal (solid) and vertical (dashed) trajectories of ϕ_0 and ψ_0 on the square torus.

homotopy classes of simple closed curves on X_0 that arise as the quotients of $\mathbb{R}$ and $i\mathbb{R}$, respectively; both γ and η have two distinct lifts ($\gamma_\pm$ and $\eta_\pm$, respectively) to X. Let α denote the separating curve on X that covers $[-1/2, 1/2]$, and let β denote the simple closed curve on X that is the union of the two lifts of $[-i, i]$. Then:

(1) the trajectories of ϕ represent (γ_+, γ_-),
(2) the trajectories of ψ represent $(\gamma_+, \gamma_-, \alpha)$, and
(3) the trajectories of both $-\phi$ and $-\psi$ represent (η_+, η_-, β).

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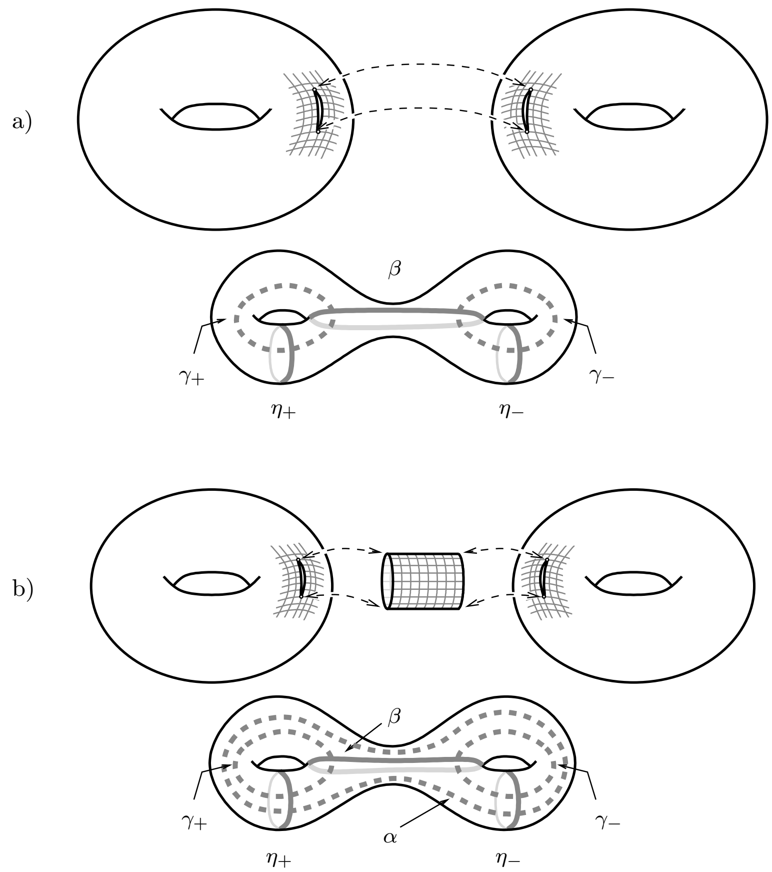

FIGURE 4. Two constructions of a surface of genus two with a Strebel differential: (a) Two tori are glued along a segment of a leaf of a Euclidean foliation. (b) Two tori are glued to the ends of a Euclidean cylinder. Below each example, the homotopy classes represented by the horizontal and vertical trajectories are shown (as solid and dashed lines, respectively). When the tori and cylinder are chosen correctly, this construction produces an example of Teichmüller geodesics that are asymptotic in one direction while converging to distinct endpoints in the opposite direction.

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Department of Mathematics, Brown University, Providence, RI 02912, USA,
E-mail address: `ddumas@math.brown.edu`