Math 549: Differentiable Manifolds I – David Dumas – Fall 2022

Midterm Exam

Due Monday, October 17, 2022 at 11:59pm

Instructions: Work on this take-home exam individually. You can consult your notes and the course texts freely. See the course syllabus for detailed exam policies.

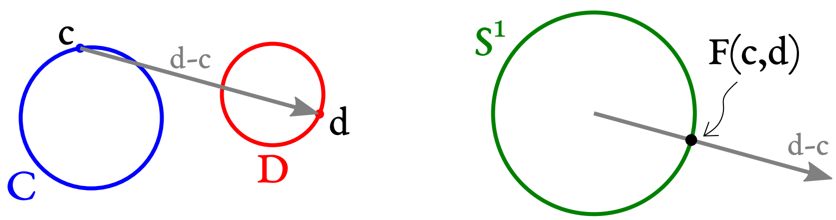
Note: After the exam is collected, I will post solutions.

(P1) **Manifold?** Let $E \subset \mathbb{R}^3$ denote the set of points defined by the equations

$$x + 2x^3 + y + y^3 + z = 6$$
$$2x^3 - y^3 = 1$$
$$x + y + 2y^3 + z = 5$$

Is E an embedded submanifold of $\mathbb{R}^3$? If so, what is its dimension? Whatever your answer, give a proof.

(P2) **The O×O problem.** Let C and D be two circles in $\mathbb{R}^2$ such that the distance between their centers is larger than the sum of the radii. We can consider C and D as manifolds, each diffeomorphic to S^1. Let $F : C \times D \to S^1$ be the map defined as follows: $F(c,d)$ is the unit vector that points in the direction of $d - c$. This construction is depicted below.



The map F is smooth (and you don't need to prove that). What are the possible values of the rank of the differential $dF_{(c,d)}$? Does it depend on the point (c,d) or it is the same everwhere? If the rank is not constant, determine where each possible value of the rank is realized.

If you prefer, you can solve this problem for a particular pair of circles C and D satisfying the given conditions. Such a solution, if correct and complete, is eligible for full credit. However, the problem admits a uniform solution for any pair of circles that doesn't involve explicit coordinate calculations.

(P3) **Reframing.** Recall that $\mathrm{GL}(2,\mathbb{R})$ is an open subset of $\mathrm{Mat}_{2\times 2}(\mathbb{R}) \simeq \mathbb{R}^4$, so it has global coordinates a,b,c,d corresponding to the entries of $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}(2,\mathbb{R})$. This means that $\frac{\partial}{\partial a}, \frac{\partial}{\partial b}, \frac{\partial}{\partial c}, \frac{\partial}{\partial d}$ is a frame for $\mathrm{GL}(2,\mathbb{R})$. However, $\mathrm{GL}(2,\mathbb{R})$ is also a Lie group, so it has a frame consisting of left-invariant vector fields. Thus it is possible to write $\frac{\partial}{\partial a}$ as a linear combination of left-invariant vector fields, where the coefficients are in $C^\infty(\mathrm{GL}(2,\mathbb{R}))$. Do so explicitly.