Math 549: Differentiable Manifolds I – David Dumas – Fall 2022

Midterm Exam Solutions

(P1) **Manifold?** Let $E \subset \mathbb{R}^3$ denote the set of points defined by the equations

$$x + 2x^3 + y + y^3 + z = 6$$
$$2x^3 - y^3 = 1$$
$$x + y + 2y^3 + z = 5$$

Is E an embedded submanifold of $\mathbb{R}^3$? If so, what is its dimension? Whatever your answer, give a proof.

Solution:

Proposition 1. *The set E is an embedded submanifold of $\mathbb{R}^3$ of dimension* 1.

Proof. Notice that the first equation is the sum of the other two. Thus E can equivalently be described as the set $F^{-1}(1,5)$ where $F : \mathbb{R}^3 \to \mathbb{R}$ is the map

$$F(x,y,z) = (2x^3 - y^3, x + y + 2y^3 + z).$$

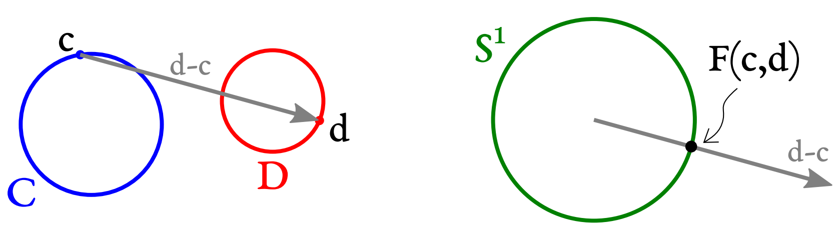
The critical points of this map are points (x,y,z) where the differential

$$DF = \begin{pmatrix} 6x^2 & -3y^2 & 0 \\ 1 & 1+6y^2 & 1 \end{pmatrix}$$

has rank less than 2. When $x \neq 0$, the first and last columns are linearly independent. When $y \neq 0$ the second and last columns are linearly independent. Thus the only critical points lie in $x = y = 0$. Since $F(0,0,z) = (0,z)$, we see that $(1,5)$ is not a critical value. Thus $F^{-1}(1,5)$ is the preimage of a regular value, hence an embedded submanifold of dimension $\dim(\mathbb{R}^3) - \dim(\mathbb{R}^2) = 1$. $\square$

Note: You could still complete the problem even if you didn't notice the linear dependence between the equations. One way to do that would be to show that $G : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $G(x,y,z) = (x + 2x^3 + y + y^3 + z, 2x^3 - y^3, x + y + 2y^3 + z)$ is a constant rank map when restricted to $\mathbb{R}^3 \setminus \{(0,0,z) \mid z \in \mathbb{R}\}$, with rank 2. Then the preimage theorem for constant rank maps applies, giving the same conclusion as above.

(P2) **The problem.** Let C and D be two circles in $\mathbb{R}^2$ such that the distance between their centers is larger than the sum of the radii. We can consider C and D as manifolds, each diffeomorphic to S^1. Let $F : C \times D \to S^1$ be the map defined as follows: $F(c,d)$ is the unit vector that points in the direction of $d - c$. This construction is depicted below.



The map F is smooth (and you don't need to prove that). What are the possible values of the rank of the differential $dF_{(c,d)}$? Does it depend on the point (c,d) or it is the same everywhere? If the rank is not constant, determine where each possible value of the rank is realized.

If you prefer, you can solve this problem for a particular pair of circles C and D satisfying the given conditions. Such a solution, if correct and complete, is eligible for full credit. However, the problem admits a uniform solution for any pair of circles that doesn't involve explicit coordinate calculations.

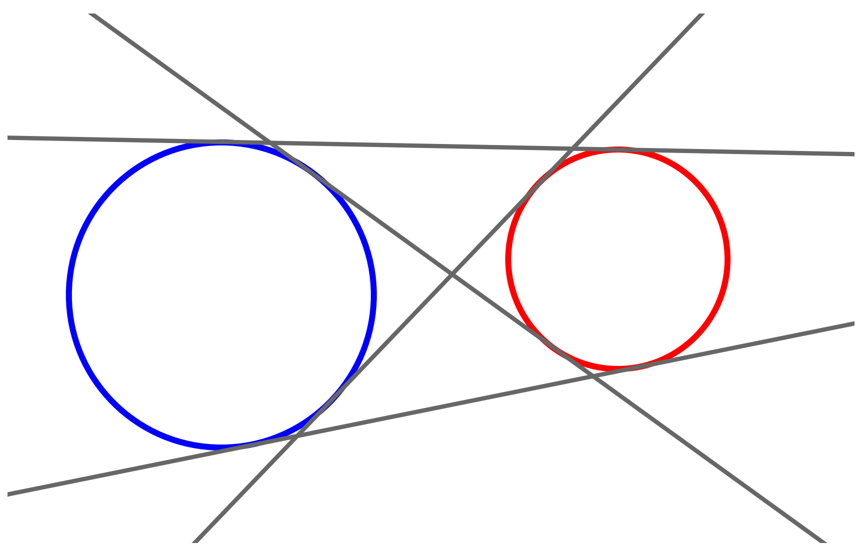
Solution: This map has 1-dimensional codomain, so the only possible values of the rank are 0 and 1, with the former corresponding to critical points of F. So the question could be rephrased in either of these equivalent forms:

- Does F have critical points, and if so, where?
- Are there points where the differential of F is zero, and if so, where?

For a pair of distinct points c,d in the plane, let $\overline{cd}$ denote the line containing them. We claim:

Proposition 2. *Both ranks* 0 *and* 1 *occur for dF. A point $(c,d) \in C \times D$ is a critical point of F if and only if the line $\overline{cd}$ is tangent to both C and D. This gives a bijection between the set of critical points of F and the set of* ***common tangent lines*** *of C and D.*

In fact one can show that there are exactly four such common tangent lines, in a configuration similar to the one shown below.



There are lots of ways to prove this proposition. To embrace the spirit of the problem's suggestion, I'll present an argument that avoids most explicit calculations. (I don't claim this is the shortest possible solution.)

Proof. It is helpful in this problem to identify $\mathbb{R}^2$ with $\mathbb{C}$. As usual when treating a vector space (or open set therein) as a smooth manifold, we can identify each tangent space with the vector space itself. That is, $T_z\mathbb{C} \simeq \mathbb{C}$ with $v \in \mathbb{C}$ corresponding to the velocity vector of the path $\gamma(t) = z + tv$ at $t = 0$.

Consider the map $N : \mathbb{C}^* \to S^1$ given by $N(z) = \frac{z}{|z|}$. This is a submersion, because the map admits local smooth sections through every point: If $z_0 = re^{i\theta}$ then the map $S^1 \to \mathbb{C}^*$, $e^{it} \mapsto re^{it}$ is a section whose image contains z_0.

The kernel of dN is therefore a one-dimensional subspace of each tangent space to $\mathbb{C}^*$. In fact $\ker dN_z$ is exactly the set of vectors that (considered as elements of $\mathbb{C}$, as noted above) are real multiples of z. This holds because these vectors form the tangent space of the embedded 1-dimensional submanifold $\{rz \,|\, r > 0\}$ of $\mathbb{C}^*$ on which N is constant.

Now let $U = \mathbb{C} \times \mathbb{C} \setminus \{(z,z) \,|\, z \in \mathbb{C}\}$ and consider the map $G : U \to \mathbb{S}^1$ given by $G(z,w) = N(w-z)$. At each point, the differential dG factors as a composition of the differential of the map $(w,z) \to w-z$ and dN. Since $(w,z) \to w-z$ is linear, it is equal to its own differential under our convention of identifying $\mathbb{C}$ with its tangent spaces. Thus $\ker dG_{(z,w)}$ is the set of vectors (s,t) such that $t-s$ lies in $\ker dN_{(w-z)}$, i.e. the set of (s,t) such that $t-s$ is in the span (over $\mathbb{R}$) of $w-z$.

Now returning to the setting of the problem, C and D are embedded submanifolds of $\mathbb{C}$, and so $C \times D$ is an embedded submanifold of $\mathbb{C} \times \mathbb{C}$. In fact $C \times D$ is contained in U, since C and D are disjoint. The map $F : C \times D \to S^1$ is the restriction of G to $C \times D$. Thus, to see whether (c,d) is a critical point of F, we must check whether the subspace $T_{(c,d)}C \times D$ of $T_{(c,d)}U$ is contained in $\ker dG$.

Letting p_C and p_D denote the centers of C and D, respectively, the tangent space $T_{(c,d)}C \times D \simeq T_cC \oplus T_dD$ consists of pairs (s,t) with s orthogonal to $c - p_C$ and t orthogonal to $d - p_D$. Thus $dF_{(c,d)}$ vanishes if and only if

$$(s \perp (c - p_C) \text{ and } t \perp (d - p_D)) \implies ((t-s) \in \operatorname{span}(d-c)) \tag{1}$$

If $\overline{cd}$ is tangent to both circles, then $d-c$ is a nonzero vector orthogonal to both $c - p_C$ and $d - p_D$ (since tangents and radii are orthogonal). Thus $c - p_C$ and $d - p_D$ both lie in the 1-dimensional space $L^\perp$, where $L = \operatorname{span}(d-c)$. Then $s \perp (c - p_C)$ and $t \perp (d - p_D)$ implies both s and t are orthogonal to $L^\perp$, hence both in L. So $s - t \in L = \operatorname{span}(d-c)$. Hence the implication above holds when $\overline{cd}$ is tangent to both circles, and such (c,d) give critical points.

Conversely, suppose the implication (1) holds. Apply this implication to $(s,t) = (s_0, 0)$ where s_0 is nonzero and orthogonal to $c - p_C$. (We can do this since $t = 0$ is trivially orthogonal to $d - p_D$.) We conclude $(t-s) = -s_0$ is in the span of $d-c$. Since s_0 is a tangent vector to C and $d-c$ is parallel to $\overline{cd}$, we've shown that $\overline{cd}$ is a tangent of C. Arguing similarly with $(s,t) = (0,t_0)$ one concludes that $\overline{cd}$ is also tangent to D.

We've now characterized critical points as pairs (c,d) with $\overline{cd}$ a common tangent line of C and D. Note that a common tangent line ℓ uniquely determines such (c,d) since $c = \ell \cap C$ and $d = \ell \cap D$.

Rank 1 is thus achieved since there exist $(c,d) \in C \times D$ with $\overline{cd}$ *not* tangent to both circles; for example take (c,d) on the line joining the centers of C and D. (As an alternative to such an explicit choice, we could argue that the image of F contains an open set in S^1, hence the image contains a regular value by Sard's theorem. Any preimage of such a regular value would be a point (c,d) where $dF_{(c,d)}$ has rank 1.)

Rank 0 is achieved since common tangents exist. A geometric way to see this is to rotate D about the center of C so that the lowest points on the two circles have the same

y coordinate. This is possible by the intermediate value theorem: The lowest point of D changes continuously under such rotation, and can be made to be either below or above the lowest point of C, because the distance between the centers is larger than the sum of the radii. After this rotation, some line of the form $y = y_0$ is a common tangent. Rotate that line back to find a common tangent of the original circles. $\square$

Optional aside: Solutions were not required to enumerate the common tangents, and we will not do so in these solutions. But for the curious I will suggest a way it might be done without explicit coordinate calculations using algebraic-geometric considerations: Think of $\mathbb{R}^2$ as an chart of $\mathbb{RP}^2$, and work in $\mathbb{RP}^2$ since the answers are simpler there. Then a circle C is a projective conic—a smooth curve defined in homogeneous coordinates by a quadratic polynomial. The set of tangent lines to a conic is a subset $C^* \subset (\mathbb{RP}^2)^*$, where $(\mathbb{RP}^2)^*$ is the space of all lines in $\mathbb{RP}^2$ (equivalently, 2-planes in $\mathbb{R}^3$). In fact, this set of tangents of a conic is *itself* a conic, i.e. also defined by a quadratic equation. So finding common tangents amounts to finding the intersections $C^* \cap D^*$ of two projective conics. If we were working over $\mathbb{C}$, then Bézout's theorem would tell us that the number of intersection points (counted with multiplicity) is $(\deg C^*)(\deg D^*) = 2 \cdot 2 = 4$. Over $\mathbb{R}$ there will in general be fewer intersections, as the real conics may have non-real intersection points (coming in complex conjugate pairs). This happens, for example, if C and D are concentric circles. But for C and D as in the problem all complex points of $C^* \cap D^*$ turn out to be real and have multiplicity one. Thus there are exactly four common tangents, projectively. Finally one must check that these correspond to lines in the $\mathbb{R}^2$ we started with and not just in $\mathbb{RP}^2$.

(P3) **Reframing.** Recall that $\mathrm{GL}(2,\mathbb{R})$ is an open subset of $\mathrm{Mat}_{2\times 2}(\mathbb{R}) \simeq \mathbb{R}^4$, so it has global coordinates a, b, c, d corresponding to the entries of $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}(2,\mathbb{R})$. This means that $\frac{\partial}{\partial a}, \frac{\partial}{\partial b}, \frac{\partial}{\partial c}, \frac{\partial}{\partial d}$ is a frame for $\mathrm{GL}(2,\mathbb{R})$. However, $\mathrm{GL}(2,\mathbb{R})$ is also a Lie group, so it has a frame consisting of left-invariant vector fields. Thus it is possible to write $\frac{\partial}{\partial a}$ as a linear combination of left-invariant vector fields, where the coefficients are in $C^\infty(\mathrm{GL}(2,\mathbb{R}))$. Do so explicitly.

Solution: The problem only asked about $\frac{\partial}{\partial a}$, but it's natural to do this for the entire frame $\frac{\partial}{\partial a}, \frac{\partial}{\partial b}, \frac{\partial}{\partial c}, \frac{\partial}{\partial d}$ at the same time.

We write a tangent vector to $\mathrm{GL}(2,\mathbb{R})$ at $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ as a matrix $\begin{pmatrix} x & y \\ z & y \end{pmatrix}$, which means the velocity vector at $t = 0$ of the path $t \mapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix} + t \begin{pmatrix} x & y \\ z & y \end{pmatrix}$ (which is in $GL(2,\mathbb{R})$ for t near 0). Equivalently $\begin{pmatrix} x & y \\ z & y \end{pmatrix}$ considered as a tangent vector at $m = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ means $x \left.\frac{\partial}{\partial a}\right|_m + y \frac{\partial}{\partial b} m + z \frac{\partial}{\partial c} m + t \frac{\partial}{\partial c} m$.

Then for $m = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, the map $L_m : \mathrm{GL}(2,R) \to \mathrm{GL}(2,R)$ is given by

$$L_m \begin{pmatrix} p & q \\ r & s \end{pmatrix} = \begin{pmatrix} ap+br & aq+bs \\ cp+dr & cq+ds \end{pmatrix}$$

This is linear in p,q,r,s, hence equal to its own differential when vectors are expressed in the the coordinate frame. In particular,

$$(dL_m)_I \begin{pmatrix} x & y \\ z & t \end{pmatrix} = \begin{pmatrix} ax+bz & ay+bt \\ cx+dz & cy+dt \end{pmatrix}.$$

(The fact that this differential is taken at the point $m = I$ did not affect the formula.) Let V^a, V^b, V^c, V^d denote the left-invariant vector fields with $(V^a)_I = \frac{\partial}{\partial a}\Big|_I$ and similarly for b,c,d. Then for $m = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ we have

$$\begin{aligned}
(V^a)_m &= (dL_m)_I \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} a & 0 \\ c & 0 \end{pmatrix} \\
(V^b)_m &= (dL_m)_I \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & a \\ 0 & c \end{pmatrix} \\
(V^c)_m &= (dL_m)_I \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} b & 0 \\ d & 0 \end{pmatrix} \\
(V^d)_m &= (dL_m)_I \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & b \\ 0 & d \end{pmatrix}
\end{aligned}$$

Equivalently we can write this as

$$\begin{aligned}
V^a &= a\frac{\partial}{\partial a} + c\frac{\partial}{\partial c} \\
V^b &= a\frac{\partial}{\partial b} + c\frac{\partial}{\partial d} \\
V^c &= b\frac{\partial}{\partial a} + d\frac{\partial}{\partial c} \\
V^d &= b\frac{\partial}{\partial b} + d\frac{\partial}{\partial d}
\end{aligned} \tag{2}$$

We can solve this system for $\frac{\partial}{\partial a}, \frac{\partial}{\partial b}, \frac{\partial}{\partial c}, \frac{\partial}{\partial d}$. We find:

$$\begin{aligned}
\frac{\partial}{\partial a} &= \frac{1}{ad-bc}\left(dV^a - cV^c\right) \\
\frac{\partial}{\partial b} &= \frac{1}{ad-bc}\left(dV^b - cV^d\right) \\
\frac{\partial}{\partial c} &= \frac{1}{ad-bc}\left(-bV^a + aV^c\right) \\
\frac{\partial}{\partial d} &= \frac{1}{ad-bc}\left(-bV^b + aV^d\right)
\end{aligned}$$

Note that $\frac{1}{ad-bc}$ is a smooth function on $\mathrm{GL}(2,R)$.

You might notice that the expressions above look like the cofactor formula for $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1}$. That's no accident! Indeed, the system of equations (2) says that the two bases $\{\frac{\partial}{\partial a}, \frac{\partial}{\partial b}, \frac{\partial}{\partial c}, \frac{\partial}{\partial d}\}$ and $\{V^a, V^b, V^c, V^d\}$ as a module over $C^\infty(\mathrm{GL}(2,\mathbb{R}))$ of $\mathrm{Vect}(M)$ are related by the change of basis matrix

$$\begin{pmatrix} a & 0 & c & 0 \\ 0 & a & 0 & c \\ b & 0 & d & 0 \\ 0 & b & 0 & d \end{pmatrix} = \begin{pmatrix} aI_{2\times 2} & cI_{2\times 2} \\ bI_{2\times 2} & dI_{2\times 2} \end{pmatrix} \in \mathrm{Mat}_{4\times 4}(C^\infty(M))$$

where in these matrices, a denotes the C^∞ function on $\mathrm{GL}(2,\mathbb{R})$ that takes a 2×2 matrix to its upper-left entry, and similarly for b, c, d. Thus the inverse of $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ (or more precisely, its transpose) shows up naturally when computing the inverse of this change of basis.

To think about: How would this problem generalize to $\mathrm{GL}(3,\mathbb{R})$? Would the last comment—about the cofactor matrix arising in the change of frame—also generalize?