Math 549: Differentiable Manifolds I – David Dumas – Fall 2022

Homework 9

Due Monday, October 31, 2022 at 11:59pm

The same instructions given on homework 1 and 2 apply.

(P1) Problem 20-5 in Lee's book, on page 536. (Explicit computation of some one-parameter subgroups of $\mathrm{GL}(3,\mathbb{R})$.)

(P2) Recall that problem (P3) on Homework 7 had you compute (among other things) a frame of left-invariant vector fields on the Lie group G that is the set $\mathbb{R}^2$ with the product $(x,y)\cdot(x',y') = (x+x', y+e^x y')$. The end result was that the vector fields

$$V^x = \frac{\partial}{\partial x} \quad \text{and} \quad V^y = e^x \frac{\partial}{\partial y}$$

are left-invariant, and hence form a basis for $\mathfrak{g}$. (Note $(V^x)_e = \frac{\partial}{\partial x}\Big|_e$ and $(V^y)_e = \frac{\partial}{\partial y}\Big|_e$.)

Compute the exponential map of this Lie group explicitly: That is, let $\exp(a,b) \in G$ denote the image of the Lie algebra element $aV^x + bV^y$ under the Lie group exponential map $\exp : \mathfrak{g} \to G$. Find an explicit expression for $\exp(a,b)$ (that is, its x and y coordinates should be explicit expressions involving only a and b).

(P3) When you have a Lie group G and a tangent vector $v \in T_eG$, you obtain a left-invariant vector field $\tilde{v}$ on G with $\tilde{v}_e = v$ in a natural way. An equivalent way to describe it is to take a path $\gamma : I \to G$ with $\gamma(0) = e$ and $\gamma'(0) = v$. Then the path $L_g \circ \gamma$ goes through g at $t = 0$ and we can define $\tilde{v}_g = (L_g \circ \gamma)'(0)$.

Now, if G acts smoothly on a manifold M, we can use a very similar idea to turn $v \in T_eG$ into a vector field $\hat{\theta}(v)$ on M: Take γ as before, then for $p \in M$ define $\hat{\theta}(v)_p = (\gamma \cdot p)'(0)$ where $\gamma \cdot p$ denotes the path in M given by $t \mapsto \gamma(t)\cdot p$.

Consider the action of $\mathrm{SL}(2,\mathbb{R})$ on $\mathbb{RP}^1$, where for $A \in \mathrm{SL}(2,\mathbb{R})$ and 1-dimensional subspace $L \subset \mathbb{R}^2$ (that is, $L \in \mathbb{RP}^1$) we define $A \cdot L = A(L) := \{Ax \mid x \in L\}$. By the construction above, for each element of $\mathfrak{sl}(2,\mathbb{R})$ we should then get a smooth vector field on $\mathbb{RP}^1$.

Compute this smooth vector field explicitly. That is, if $x \in \mathfrak{sl}(2,\mathbb{R})$ is given by $ae + bh + cf$ for $a,b,c \in \mathbb{R}$ and

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

then compute the vector field $\hat{\theta}(x) \in \mathrm{Vect}(\mathbb{RP}^1)$.

(Actually, it suffices to compute $\hat{\theta}(x)$ in one of the charts of $\mathbb{RP}^1$, since these charts cover all but one point, and knowing a vector field everywhere except one point uniquely determines it at that point by continuity. So I'll accept an expression in one affine chart.)