Math 549: Differentiable Manifolds I – David Dumas – Fall 2022

Homework 7

Due Monday, October 10, 2022 at 11:59pm

The same instructions given on homework 1 and 2 apply.

(P1) Suppose M is a connected manifold. Show that for any $p,q \in M$ there exists a diffeomorphism $F : M \to M$ with $F(p) = q$.

Suggested outline (each step requires justification):

- For $p,q \in M$, let's say $p \sim q$ if there exists a diffeomorphism $F : M \to M$ with $F(p) = q$. Then this is an equivalence relation.
- For any q in the unit ball in $\mathbb{R}^n$, the vector field $\sum_i q_i \frac{\partial}{\partial x_i}$ has an integral curve that contains both 0 and q.
- For any q in the unit ball in $\mathbb{R}^n$, there is a vector field supported in $\{x : |x| \leq 1-\varepsilon\}$ and which has an integral curve that contains both 0 and q.
- The equivalence classes of the equivalence relation introduced above are **open** subsets of M.
- Use connectedness.

(P2) Lee, Chapter 7, Problem 2, on page 171. (Differentials of m and i)

(P3) Consider the binary operation on $\mathbb{R}^2$ defined by the formula

$$(x,y) \cdot (x',y') = (x+x', y+e^x y').$$

(a) Show that this gives a Lie group structure on $\mathbb{R}$ with identity element $e = (0,0)$. For the rest of this problem, let G denote this Lie group.

(b) Show that G is not isomorphic (as a Lie group) to $(\mathbb{R}^2,+)$.

Then, for each vector field listed below, find both a formula for it in terms of $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$, and draw a sketch of it in the xy plane.

(c) The extension of $\left.\frac{\partial}{\partial x}\right|_e$ to a left-invariant vector field.

(d) The extension of $\left.\frac{\partial}{\partial x}\right|_e$ to a right-invariant vector field.

(e) The extension of $\left.\frac{\partial}{\partial y}\right|_e$ to a left-invariant vector field.

(f) The extension of $\left.\frac{\partial}{\partial y}\right|_e$ to a right-invariant vector field.