Math 549: Differentiable Manifolds I – David Dumas – Fall 2022

Homework 6

Due Monday, October 3, 2022 at 11:59pm

The same instructions given on homework 1 and 2 apply.

(P1) For any $n \geq 1$, construct a smooth vector field on $\mathbb{S}^n$ that vanishes at exactly one point.

Hint: Use the stereographic projection charts.

(P2) Let $\mathrm{SL}_2\,\mathbb{R}$ denote the set of 2×2 matrices of determinant 1, which is a smooth manifold of dimension 3. (For example, it is the preimage $\det^{-1}(1)$, and 1 is a regular value of $\det : \mathrm{Mat}_{2\times 2}\mathbb{R} \to \mathbb{R}$.) Recall that the product of two matrices of determinant 1 also has determinant 1, which will be helpful in this problem.

(a) Let I denote the 2×2 identity matrix. Here are three smooth paths $\mathbb{R} \to \mathrm{SL}_2\,\mathbb{R}$ that go through I at $t = 0$:

- $\gamma_1(t) = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$
- $\gamma_2(t) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$
- $\gamma_3(t) = \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}$

Show that their velocity vectors, $\{\gamma_i'(0) \mid 1 \leq i \leq 3\}$, form a basis of $T_I\,\mathrm{SL}_2\,\mathbb{R}$.

(b) Let $m = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2\,\mathbb{R}$. Construct an explicit basis of $T_m\,\mathrm{SL}_2\,\mathbb{R}$.

Suggestion: Use part (a) and the matrix multiplication operation on $\mathrm{SL}_2\,\mathbb{R}$.

(P3) As in the previous problem, $\mathrm{SL}_2\,\mathbb{R}$ denotes the 3-dimensional manifold consisting of all 2×2 matrices of determinant 1.

(a) Let $\mathrm{tr} : \mathrm{SL}_2\,\mathbb{R} \to \mathbb{R}$ denote the trace map, which takes a matrix to the sum of its diagonal entries, i.e.

$$\mathrm{tr}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = a + d$$

(But note we've restricted attention to matrices of determinant 1.)

What are the regular values of this function?

(b) Suppose $\lambda > 1$. Show that the set of matrices

$$\{A \in \mathrm{SL}_2\,\mathbb{R} \mid \lambda \text{ is an eigenvalue of A}\}$$

is an embedded submanifold of $\mathrm{SL}_2\,\mathbb{R}$. What is its dimension?