Math 549: Differentiable Manifolds I – David Dumas – Fall 2022

Homework 4

Due Monday, September 19, 2022 at 11:59pm

The same instructions given on homework 1 and 2 apply.

(P1) Let $\pi : (\mathbb{R}^{n+1} \setminus \{0\}) \to \mathbb{RP}^n$ be the quotient map.

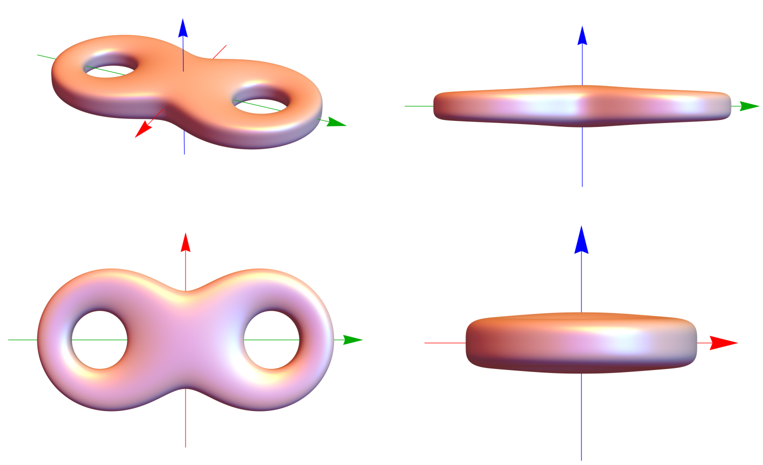
(a) Is π an immersion? A submersion?

(b) Same questions for the map $\pi|_{\mathbb{S}^n} : \mathbb{S}^n \to \mathbb{RP}^n$ (i.e. the quotient map restricted to the unit sphere)

(P2) Let M be a smooth manifold of dimension n. Then $M \times M$, which is a smooth manifold of dimension $2n$, is the set of all ordered pairs of points in M. The subset $\Delta_M := \{(p,p) \mid p \in M\} \subset M \times M$ is closed, so $M \times M \setminus \Delta_M$ is an open subset of $M \times M$ and thus a smooth manifold as well.

Let $B_2(M)$ denote the set of all 2-element subsets of M. (Equivalently, $B_2(M)$ is the set of all *unordered pairs* of distinct points in M.) Construct a smooth structure on $B_2(M)$ that makes it a manifold of dimension $2n$ in such a way that the map $\sigma : (M \times M \setminus \Delta_M) \to B_2(M)$ given by $\sigma((p,q)) = \{p,q\}$ is a smooth covering map.

(P3) Construct a polynomial $P(x,y,z)$ such that $P^{-1}(0)$ looks as much like the picture below as you can manage, and so that $P : \mathbb{R}^3 \to \mathbb{R}$ is a submersion when restricted to an open set U that contains $P^{-1}(0)$. Explain what you consider to be the essential features of the picture that your example realizes, and why it does so.



Note this is four views of a single surface in $\mathbb{R}^3$.

Note: In class I said I would ask you to prove that $\pi \circ r_{\sqrt{2}} : \mathbb{R} \to \mathbb{T}^2$ has dense image. That's a nice problem, but I decided against including it on this assignment.