Math 549: Differentiable Manifolds I – David Dumas – Fall 2022

Homework 3

Due Monday, September 12, 2022 at 11:59pm

The same instructions given on homework 1 and 2 apply.

(P1) Let $V = \{(x,y) \mid y = |x|\} \subset \mathbb{R}^2$.
  (a) Construct a smooth map $f : \mathbb{R} \to \mathbb{R}^2$ such that $f(\mathbb{R}) = V$.
  (b) Show that there is no immersion $f : \mathbb{R} \to \mathbb{R}^2$ such that $f(\mathbb{R}) = V$. (Do this "directly", without appealing to results we'll discuss later such as the implicit function theorem.)

(P2) Show that the map $f : \mathbb{RP}^2 \to \mathbb{R}^6$ defined by $f([x,y,z]) = \frac{1}{x^2+y^2+z^2}(x^2, y^2, z^2, yz, xz, xy)$ is a smooth embedding.

Extension of this problem (don't turn in): Can you find an embedding $\mathbb{RP}^2 \to \mathbb{S}^5$?

For the topological part of (P2), the following result (which you do not need to prove) may be useful:

Theorem. *Let X be a compact topological space. Let Y be a Hausdorff topological space. If $f : X \to Y$ is a continuous bijection, then f is a homeomorphism.*

Corollary. *If $f : X \to Y$ is a continuous injection, where X is compact and Y is Hausdorff, then f is a topological embedding.*

(These are stated as parts of Lemma A.52 in Lee's book, where references to proofs can also be found. The key to the proof is that compact subsets of Hausdorff spaces are closed.)

Revision history:

- 2022-09-11 Fixed a typo in problem P2; as originally stated, the map was not well-defined.