Math 549: Differentiable Manifolds I – David Dumas – Fall 2022

Homework 2

Due Wednesday, September 7, 2022 at 11:59pm

Instructions:

(i) Write solutions with complete sentences that explain your answer. Don't just write a series of formulas and symbols.
(ii) Write as if the audience is another student in the class.
(iii) If a problem asks for a proof, begin it with"Proof:" and end with □ or QED.
(iv) Submit your solutions to Gradescope. (Do not give paper to course staff.)
(v) Do not include this document as part of what you submit.
(vi) Label solutions with the problem numbers from the list below, e.g. "(P3)". If a problem comes from the textbook, you can also include its number from the book if you want, but this is optional whereas the P-number is required.

Attribution notes:

- If a problem is marked [ND], it means that it was based on a problem written by Nathan Dunfield for a 2014 manifolds course at UIUC.
- If a problem is marked [LEE], it means it is a modified version of an exercise in Lee's book.

(P1) Suppose M is a smooth manifold, $p \in M$, and (U, φ) is a coordinate chart with $p \in U$. Show that for any smooth function $f_0 : \mathbb{R}^n \to \mathbb{R}$, there exists a smooth function $f : M \to \mathbb{R}$ so that $f_0 \circ \varphi$ and f are equal on an open neighborhood of p. [ND]

Suggested approach:

(a) Read Lemma 2.22 in Lee and use a bump function like the ones constructed there to build a function f_1 on $\mathbb{R}^n$ that is equal to f_0 near $\varphi(p)$ but whose support[1] is contained in $\varphi(U)$.
(b) Make f zero on most of the manifold and equal to $f_1 \circ \varphi$ on U. Then show this is smooth.

(P2) Let M_1, M_2 be smooth manifolds. In Chapter 1, Lee describes how the product space $M_1 \times M_2$ has a natural smooth manifold structure (using charts that are products of charts of M_1 and M_2). This problem will use that smooth structure.

For $i = 1, 2$ let $\pi_i : M_1 \times M_2 \to M_i$ denote the projection map $(p_1, p_2) \mapsto p_i$. This is a smooth map (Lee, Example 2.13). Show that the map

$$\alpha : T_{(p_1,p_2)}(M_1 \times M_2) \to T_{p_1}M_1 \oplus T_{p_2}M_2$$

defined by

$$\alpha(v) = (d(\pi_1)_p(v), d(\pi_2)_p(v))$$

is an isomorphism. [LEE]

[1] As defined on Page 43 of Lee, the support of a function f is the closure of the set of points where f is not equal to zero.

(P3) This problem develops another perspective on tangent vectors. First, define a *curve through* $p \in M$ to be a smooth map $\gamma : I \to M$ with $\gamma(0) = p$, where I is an interval in $\mathbb{R}$ containing 0. Note that if γ is a curve through p, and $f \in C^\infty(M)$, then $f \circ \gamma$ is a smooth function on an interval in $\mathbb{R}$, so we can take its derivative $(f \circ \gamma)'$ in the usual sense. This even works if f is a smooth function that is only defined on a neighborhood of p, but in that case we may need to shrink I a bit so that $\gamma(I)$ is contained in the domain of f.

Two curves γ_1, γ_2 through $p \in M$ are said so have the *same velocity at p* if

$$(f \circ \gamma_1)'(0) = (f \circ \gamma_2)'(0)$$

for all smooth functions f defined on a neighborhood of p. It's easy to see that this is an equivalence relation on the set of curves through p, so we'll denote it by $\sim$.

Therefore we can define the *set of velocities at p* as:

$$\mathscr{V}_pM = \{\text{smooth curves through } p\}/\sim$$

(a) Show if $[\gamma] \in \mathscr{V}_pM$ then the map $C^\infty(M) \to \mathbb{R}$ given by $f \mapsto (f \circ \gamma)'(0)$ is a derivation of $C^\infty(M)$ at p.
(b) Show that the map $R : \mathscr{V}_pM \to T_pM$ given by $R([\gamma])(f) = (f \circ \gamma)'(0)$ is a bijection.

Conclusion: A tangent vector to M at p can also be interpreted as an equivalence class of curves through p. [LEE]

Here's something to think about after doing (P3): There's no canonical way to take the sum of two curves through p, nor a scalar multiple of a curve through p. If you try to do this, you will almost certainly build something that depends on the chart you use to make the construction, with different charts giving different results. Yet T_pM is a vector space, and we've just shown its elements can be thought of as curves up to an equivalence relation. So while curves cannot be added, equivalence classes can be.