Math 549: Differentiable Manifolds I – David Dumas – Fall 2022

Homework 11

Due Monday, November 14, 2022 at 11:59pm

The same instructions given on homework 1 and 2 apply.

(P1) Let $(e_i)_{i=1\dots4}$ denote the standard basis of $\mathbb{R}^4$ and $(\omega_i)_{i=1\dots4}$ its dual basis. Let $f_i = \sum_{k=1}^{i} e_k$, so that $(f_i)_{i=1\dots4}$ is also a basis of $\mathbb{R}^4$. Let $(\eta_i)_{i=1\dots4}$ denote the dual basis of $(f_i)_{i=1\dots4}$.

(a) Write η_k as a linear combination of $(\omega_i)_{i=1\dots4}$.

(b) For any i,j write $\eta_k \wedge \eta_\ell \in \bigwedge^2(\mathbb{R}^4)^*$ as a linear combination of $(\omega_i \wedge \omega_j)_{1\le i<j\le 4}$.

(c) For any vector $v \in \mathbb{R}^4$ we can form an element $\theta_v \in \bigwedge^3(\mathbb{R}^4)^*$ by defining $\theta_v(a,b,c) = \det(v,a,b,c)$. Does there exist a vector v so that $\theta_v = \eta_2 \wedge \eta_3 \wedge \eta_4$? Either find such v or show none exists.

(P2) Using the inclusion of S^2 into $\mathbb{R}^3$, we can identify T_pS^2 with the subspace of $\mathbb{R}^3$ consisting of vectors that are orthogonal to p. Using this identification, consider the 2-form $\omega \in \Omega^2(S^2)$ defined by $\omega_p(v,w) = \det(p,v,w)$. Find an explicit expression for ω in the coordinates (u,v) given by stereographic projection $S^2 \setminus \{(0,0,1)\} \to \mathbb{R}^2_{u,v}$. (That is, the final answer should only involve du, dv, and functions of u and v.)

(P3) Problem 14-7 in Lee (p375) asks you to compute $F^*(\omega)$ and $d\omega$ in several examples. Do the half of this problem that involves calculating $F^*(\omega)$. (The $d\omega$ part might be assigned next time.)

Revision history:

- 2022-11-12 Change subscript of e in definition of f_i so the summands are not all the same vector.