Math 549: Differentiable Manifolds I – David Dumas – Fall 2022

Homework 10

Due Monday, November 7, 2022 at 11:59pm

The same instructions given on homework 1 and 2 apply. As before, attribution note [ND] means a problems derived from Nathan Dunfield's course materials.

(P1) Problem 11-7 from Lee, on page 300. (Computing some explicit pullbacks.)

(P2) Let G be a Lie group with Lie algebra $\mathfrak{g}$. Recall that $\mathfrak{g}$ is the subalgebra of $\mathrm{Vect}(G)$ consisting of left-invariant vector fields. Show that the following vector spaces are naturally isomorphic to one another, and describe the isomorphisms explicitly:

- $\mathfrak{g}^* = \mathrm{Hom}(\mathfrak{g}, \mathbb{R})$, the dual space of the vector space $\mathfrak{g}$.
- The subspace of $\Omega^1(G)$ consisting of left-invariant 1-forms, i.e. the space of all 1-forms ω such that $L_g^*\omega = \omega$ for all $g \in G$.
- The cotangent space at the identity, T_e^*G.

(P3) Let $G = \mathrm{GL}(2,\mathbb{R})$, and let $\omega_{i,j} \in \Omega^1(G)$ denote the left-invariant 1-form on G which corresponds, by the isomorphism constructed in the previous problem, to the element of $T_e^*G \simeq \mathrm{Hom}(\mathrm{Mat}_{2\times 2}(\mathbb{R}), \mathbb{R})$ given by $A \mapsto A_{i,j}$ (the linear function taking a matrix to its i, j entry).

Compute $\displaystyle\int_\gamma \omega_{i,j}$ for each of the following parametrized curves in G:

(a) $\gamma(t) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$ for $t \in [0,1]$.

(b) $\gamma(t) = \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix}$ for $t \in [0,1]$.

(c) $\gamma(t) = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}$ for $t \in [0,2\pi]$.

[ND]