MATH 549 FALL 2022 - FINAL EXAM SOLUTIONS — David Dumas

(P1) Consider the vector field $V = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} - 2z\frac{\partial}{\partial z}$ on $\mathbb{R}^3$.

(a) Compute Vf where $f(x,y,z) = x + y^2$

(b) Compute the Lie bracket $[V, \frac{\partial}{\partial z}]$.

(c) Compute the flow of V.

(a) $$Vf = \left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} - 2z\frac{\partial}{\partial z}\right)\overbrace{(x+y^2)}^{\text{independent of } z}$$

$$= x\frac{\partial}{\partial x}(x+y^2) + y\frac{\partial}{\partial y}(x+y^2) = x(1) + y(2y)$$

$$= \boxed{x + 2y^2}$$

(b) We could use the general formula, but applying the definition directly is probably just as efficient.

If $[V, \frac{\partial}{\partial z}] = a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y} + c\frac{\partial}{\partial z}$

then $a = [V, \frac{\partial}{\partial z}]x$, $b = [V, \frac{\partial}{\partial z}]y$, $c = [V, \frac{\partial}{\partial z}]z$.

So

$$a = V\left(\frac{\partial}{\partial z}x\right) - \frac{\partial}{\partial z}(Vx) = 0 - \frac{\partial}{\partial z}(x) = 0$$

$$b = V\left(\frac{\partial}{\partial z}y\right) - \frac{\partial}{\partial z}(Vy) = 0 - \frac{\partial}{\partial z}(y) = 0$$

$$c = V\left(\frac{\partial}{\partial z}z\right) - \frac{\partial}{\partial z}(Vz) = \underbrace{V(1)}_{\substack{\text{const} \\ 0}} - \frac{\partial}{\partial z}(-2z) = 2$$

So $\boxed{[V, \frac{\partial}{\partial z}] = 2\frac{\partial}{\partial z}}$

P1 continued

(c) Claim: The flow is given by $\Theta_t(x,y,z)=(e^t x, e^t y, e^{-2t} z)$.

<u>Proof</u>. Since $\Theta_0(x,y,z)=(x,y,z)$ we need only show that for any (x_0,y_0,z_0), the curve

$$\gamma(t)=\Theta_t(x_0,y_0,z_0)=(e^t x_0, e^t y_0, e^{-2t} z_0)$$

is an integral curve of V. This follows from the equality of

$$\gamma'(t)=e^t x_0 \frac{\partial}{\partial x}+e^t y_0 \frac{\partial}{\partial y}-2e^{-2t} z_0 \frac{\partial}{\partial z}$$

and

$$V_{\gamma(t)}=V_{(e^t x_0, e^t y_0, e^{-2t} z_0)}=e^t x_0 \frac{\partial}{\partial x}+e^t y_0 \frac{\partial}{\partial y}-2e^{-2t} z_0 \frac{\partial}{\partial z}$$

(P2) Consider these differential forms on $\mathbb{R}^3$:

$$\alpha = dy \wedge dz + z\,dx \wedge dz$$
$$\beta = x\,dy - y\,dx$$

Also, let $\gamma : \mathbb{R} \to \mathbb{R}^3$ be the map $\gamma(t) = (t, t^2, 0)$.

(a) Compute $\alpha \wedge \beta$.

(b) Compute $d\alpha$.

(c) Compute $d\beta$.

(d) Compute $\gamma^*(\alpha)$.

(e) Compute $\gamma^*(\beta)$.

(a) $\alpha \wedge \beta = (dy \wedge dz + z\,dx \wedge dz) \wedge (x\,dy - y\,dx)$

$$\left.\begin{aligned} &= x\,dy\wedge dz \wedge dy - y\,dy \wedge dz \wedge dx \\ &\quad + xz\,dx \wedge dz \wedge dy - yz\,dx \wedge dz \wedge dx \end{aligned}\right\}\ C^\infty(M)\text{-bilinearity of } \wedge$$

$= -y\,dy \wedge dz \wedge dx + xz\,dx \wedge dz \wedge dy$

Remove terms with repeated dx_i (vanish by antisymmetry)

$= +y\,dy \wedge dx \wedge dz - xz\,dx \wedge dy \wedge dz$ swap dx_i

$= \boxed{(-y - xz)\,dx \wedge dy \wedge dz}$ Swap and group.

(b) $d\alpha = d(dy \wedge dz + z\,dx \wedge dz) = 0 + 1 \cdot \cancel{dz \wedge dx \wedge dz}$
$= \boxed{0}$

(c) $d\beta = d(x\,dy - y\,dx) = dx \wedge dy - dy \wedge dx = \boxed{2\,dx \wedge dy}$

(d) α is a 2-form and $\mathbb{R}$ is a 1-manifold, so $\boxed{\gamma^*\alpha = 0.}$

P2 continued

(e) $\gamma(t) = (t, t^2, 0)$.

$$\gamma'(t) = d\gamma\left(\frac{\partial}{\partial t}\right) = \frac{\partial}{\partial x} + 2t\frac{\partial}{\partial y}.$$

So $\gamma^*\beta = f\,dt$ where $f(t) = \beta_{(t,t^2,0)}\left(\frac{\partial}{\partial x} + 2t\frac{\partial}{\partial y}\right)$

$$= (t\,dy - t^2 dx)\left(\frac{\partial}{\partial x} + 2t\frac{\partial}{\partial y}\right)$$

$$= 2t^2 - t^2 = t^2.$$

That is, $\boxed{\gamma^*\beta = t^2 dt}$

(P3) Suppose M and X are smooth manifolds and that there are three smooth maps

$$A : M \to X,$$
$$B : X \to X,$$
$$C : X \to M$$

such that $F = C \circ B \circ A : M \to M$ is a diffeomorphism. Does this imply that any of the maps A, B, C is an immersion? And does it imply that any of A, B, C is a submersion? Justify your answers.

In addition to any arguments, examples, or justifications you write, please include a table like this one that summarizes your answer

	A	B	C
Must be an immersion?			
Must be a submersion?			

writing "yes" or "no" in each box as appropriate.

Summary

	A	B	C
Must be immersion?	YES	NO	NO
Must be submersion?	NO	NO	NO

$F = C \circ B \circ A$ is both an immersion and a submersion (as it is a diffeo)

Prop. A must be an immersion.

Proof. Suppose not. Then $dA(v) = 0$ for some nonzero vector $v \in T_pM$. But then $dF(v) = dC(dB(dA(v))) = 0$, and F is not an immersion. Contradiction. $\square$

An example where B, C not immersions and A, B, C not submersions:

$M = \mathbb{R}$ $X = \mathbb{R}^3$ $A(t) = (t, 1, 0)$ $B(x, y, z) = (x, y, 0)$

$C(x, y, z) = xy.$

Here, $C \circ B \circ A = C(B(t,1,0)) = C(t,1,0) = t \cdot 1 = t$

So $F = Id_{\mathbb{R}}$.

B not immersion: $dB\left(\frac{\partial}{\partial z}\right) = 0$

C not immersion: follows for dimension reasons

A not submersion: follows for dimension reasons

B not submersion: $dB(T_p\mathbb{R}^3)$ does not contain $\frac{\partial}{\partial z}\Big|_{B(p)}$

$(\text{rank } dB_p = 2 < \dim T_{B(p)}\mathbb{R}^3)$

C not submersion: $C(0,0,0) = 0$

but $dC_{(0,0,0)} : T_{(0,0,0)}\mathbb{R}^3 \to T_0\mathbb{R}$

is the zero map because $\left(\frac{\partial C}{\partial x}, \frac{\partial C}{\partial y}, \frac{\partial C}{\partial z}\right)$ vanish at $x = y = z = 0$.

in particular not surjective. $\square$

<u>Discussion</u>: The reason A is an immersion is that if dA sends a vector to zero, then there's no "recovering" from that. $dF = dC \circ dB \circ dA$ will also do so.

P3 Continued.

But for the maps B and C, knowing things about $C \circ B \circ A$ will only tell us about their behavior at points in the image of A or $B \circ A$, respectively. There's nothing in the problem suggesting A and $B \circ A$ are surjective, so we have no control over B and C at certain points.

To wit: "B is an immersion" means $\forall q \in X, \dots$ and the best we could hope for in the situation of this problem would be a statement like $\forall q \in A(M) \subset X, \dots$

I chose an example above where M and X have different dimensions, but the conclusion would be almost the same if the problem added the hypothesis that $\dim M = \dim X$. You could make sure A is not surjective, B misbehaves (e.g. has dB vanish at a point) somewhere in $X \setminus A(M)$ and is not surjective, and that C misbehaves somewhere in $X \setminus B(X)$.

The only thing that would change if $\dim M = \dim X$ was added is that A would be a submersion.

This problem tried to highlight the domain-centric character of the notions of immersion and submersion.