

# LECTURE 27

## JULIA SETS

MCS 275 Spring 2021

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# LECTURE 27: JULIA SETS

Course bulletins:

- [Project 3](#) due 6:00pm CDT on Friday March 19.
- Some time today:
  - Project 3 autograder opens
  - Worksheet 10

# PLAN

- Finish `numpy` introduction notebook
- `Julia sets` notebook
- `Julia sets` notebook #2

# DYNAMICS

Start with a function, e.g.  $f(x) = x^2$ , and a number,  $a$ .

Apply  $f$  over and over again to get a sequence:

$$a, f(a), f(f(a)), f(f(f(a))), \dots$$

This sequence  $\{f^n(a)\}_{n \geq 0}$  is the **orbit** of  $a$  (under  $f$ ).

What can we say about it?

Let's start with a simple example,  $f(x) = x^2$ .

$$2 \rightarrow 4 \rightarrow 16 \rightarrow 256 \rightarrow 65536 \rightarrow \dots$$

$$\frac{1}{2} \rightarrow \frac{1}{4} \rightarrow \frac{1}{16} \rightarrow \frac{1}{256} \rightarrow \frac{1}{65536} \rightarrow \dots$$

What happens for other starting points?

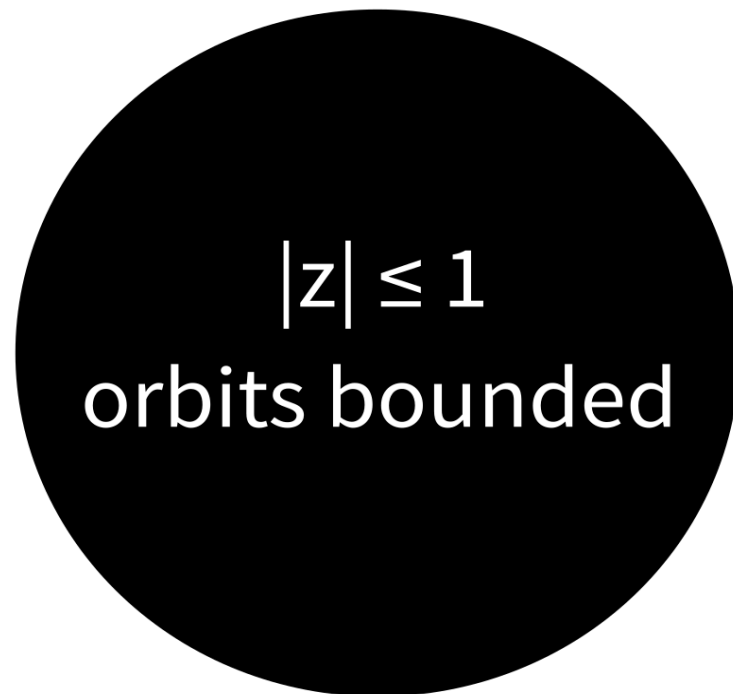
For real numbers, orbits under  $f(x) = x^2$  are easy:

- If  $|a| < 1$ , the orbit of  $a$  converges to 0
- If  $|a| > 1$ , the orbit of  $a$  goes to  $\infty$
- If  $|a| = 1$  the orbit is bounded but doesn't go to 0

For  $f(x) = x^2$  with complex numbers, it's the same!

- If  $|a| < 1$ , the orbit of  $a$  converges to 0
- If  $|a| > 1$ , the orbit of  $a$  goes to  $\infty$
- If  $|a| = 1$  the orbit is bounded but doesn't go to 0

$$f(x) = x^2$$



$|z| > 1$  orbits unbounded

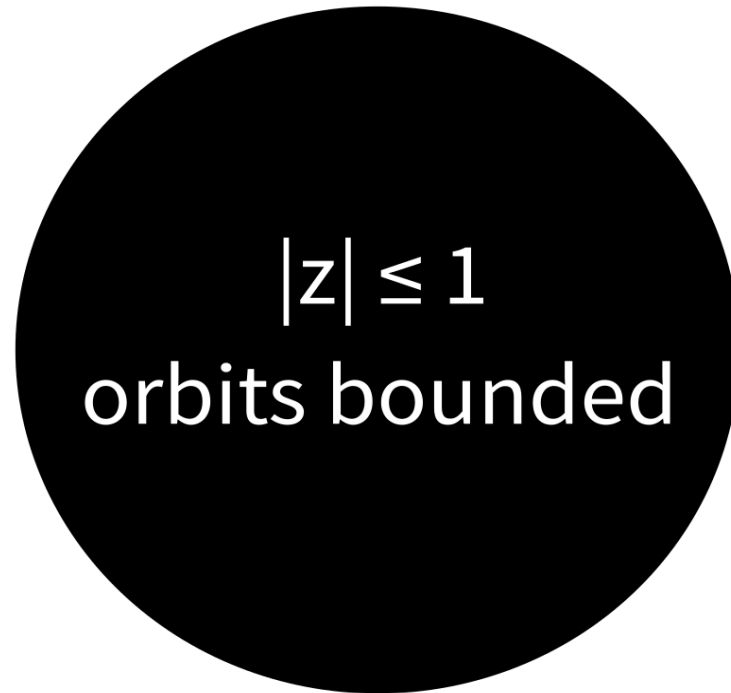


# FILLED JULIA SET

The **filled Julia set** of a polynomial  $f(x)$ , denoted  $K_f$ , is the set of complex numbers that have bounded orbits under  $f$ .

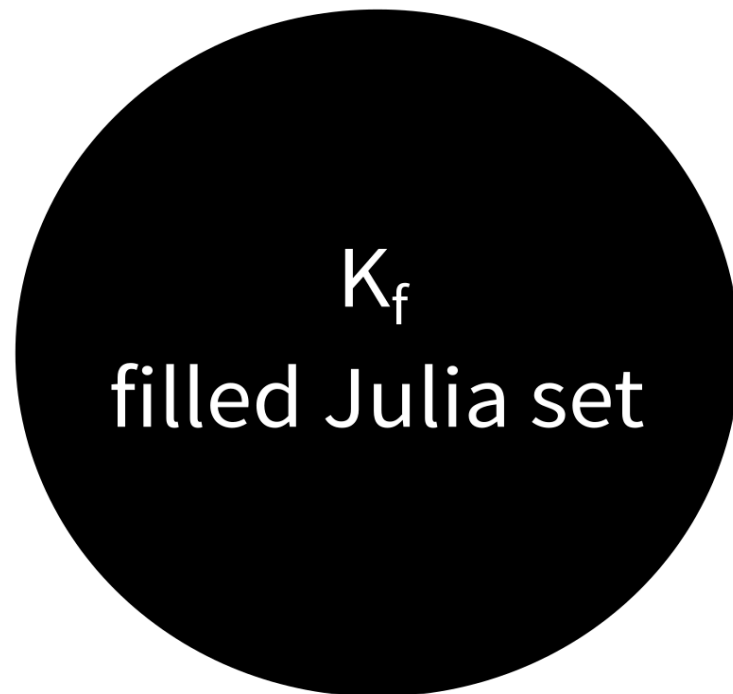
Named for mathematician Gaston Julia (1893-1978) who studied these sets starting in the 1920s.

$$f(x) = x^2$$



$|z| > 1$  orbits unbounded

$$f(x) = x^2$$



Next, let's look at  $f(x) = x^2 - 1$ .

# FIRST DESIGN

- Write a function to test orbit behavior of a point.
- Make a grid of points in  $\mathbb{C}$  and test each one.
- Record the results in an image file (black=bounded, white=unbounded).

# DIVERGENCE CRITERION

For  $f(x) = x^2 - 1$ , if  $|a| \geq 2$ , then  $f^n(a) \rightarrow \infty$  as  $n \rightarrow \infty$ .

# IMPROVED DESIGN

- Make a grid of points in  $\mathbb{C}$  and test each one.
- Use numpy array operations to apply  $f$  to all of them at once, repeatedly.
- Optional: Keep track of ones that are already big, and don't apply  $f$  to them.
- Record the results in an image file (black=bounded, white=unbounded).

# REFERENCES

- [Python Data Science Handbook by Jake VanderPlas](#)
  - [Chapter 2](#) contains the introduction to numpy.
  - There is also a print edition from O'Reilly.
- Roland Roeder's lecture notes [Around the boundary of complex dynamics](#) would be a good place for undergraduates to learn more about complex dynamics. (The most popular texts on this topic assume much more background in complex analysis and would be less accessible to an undergraduate audience.)

# REVISION HISTORY

- 2021-03-17 Add second Julia notebook link
- 2021-03-15 Add Julia notebook link
- 2021-03-14 Initial publication



