Theorem 1. *Let Γ be a group that acts by homeomorphisms on spaces X and F, and where the action on X is free and properly discontinuous. Let Γ act on $X \times F$ by $\gamma \cdot (x, f) = (\gamma \cdot x, \gamma \cdot f)$. Then $(X \times F)/\Gamma \to X/\Gamma$ by $[(x, f)] \mapsto [x]$ is a fiber bundle with fiber F. Furthermore, this fiber bundle has a natural Γ-structure (where Γ is given the discrete topology).*

The main fact about properly discontinuous group actions that we will use is that they give rise to regular covers. Specifically, if $b \in X/\Gamma$ and if $\pi : X \to X/\Gamma$ is the quotient map, then there is a neighborhood U of b such that $\pi^{-1}(U) = \bigcup_{\alpha \in J} V_\alpha$ where each V_α is open in X, the restriction $\pi|_{V_\alpha} : V_\alpha \to U$ is a homeomorphism, and where the action of Γ on X permutes the sets V_α by a transitive and free action on the index set J. We refer to U as in this condition as an *evenly covered neighborhood* of b, the sets V_α are the *sheets* over U, and the inverses $\left(\pi|_{V_\alpha}\right)^{-1} : U \to V_\alpha$ are the *sheet maps*.

Furthermore, if $\sigma : U \to X$ is a continuous right inverse of π, for $U \subset X/\Gamma$ open, and if $b \in U$, then there exists an evenly covered neighborhood U' of b such that σ is equal to one of the sheet maps for that neighborhood. (One can take $U' = \sigma^{-1}(V_\alpha)$ where V_α is the sheet containing $\sigma(b)$.)

Proof. It is easy to see that $[(x, f)] \mapsto [x]$ is a well-defined map. We denote this map by p. The map p is continuous because it is the quotient of the continuous and Γ-equivariant map $\pi_1 : X \times F \to X$.

Now we construct local trivializations. Let $b \in X/\Gamma$, let U be an evenly covered neighborhood of b and let $\sigma : U \to V_\alpha$ be one of the sheet maps. We define a map

$$\Phi : U \times F \to (X \times F)/\Gamma$$
$$\Phi(u, f) = [(\sigma(u), f)].$$

By definition this map has the form $\Phi = \pi_{X \times F} \circ \tilde{\Phi}$ where $\tilde{\Phi} : U \times F \to X \times F$ is given by $\sigma \times \mathrm{id}_F$ and where $\pi_{X \times F} : X \times F \to (X \times F)/\Gamma$ is the quotient map. Note that $\tilde{\Phi}$ is in fact a homeomorphism onto $V_\alpha \times F$.

Since σ and π are continuous, we find Φ is continuous as well. The image of Φ is contained in $p^{-1}(U)$ because σ is right inverse to π, and is equal to $p^{-1}(U)$ because if $[(x, f)] \in p^{-1}(U)$ and if we define $u = \pi(x)$, then we have $\sigma(u) = \gamma \cdot x$ for a unique $\gamma \in \Gamma$ and

$$\Phi(u, \gamma \cdot f) = [(\sigma(u), \gamma \cdot f)] = [(\gamma \cdot x, \gamma \cdot f)] = [\gamma \cdot (x, f)] = [(x, f)].$$

It follows similarly that Φ is injective: If $\Phi(u, f) = \Phi(u', f')$ then $u = \pi(\Phi(u, f)) = \pi(\Phi(u', f')) = u'$, and so $[(\sigma(u), f)] = \Phi(u, f) = \Phi(u, f') = [(\sigma(u), f')]$. That is, there exists $\gamma \in \Gamma$ so that $(\sigma(u), f) = \gamma \cdot (\sigma(u), f')$. Since Γ acts freely and transitively on the fibers of π, we have $\gamma = e$ and $f = f'$.

Finally, Φ is an open map: It suffices to check that $\Phi(U' \times W)$ is open for U' open in U and W open in F. Since $\tilde{\Phi}$ is a homeomorphism onto $V_\alpha \times F$, we have that $\tilde{\Phi}(U' \times W)$ is open in $V_\alpha \times F$. Thus $Z = \bigcup_{\gamma \in \Gamma} \gamma \cdot \tilde{\Phi}(U' \times W)$ is a saturated open set. Since $\pi_{X \times F}(Z) = \Phi(U' \times W)$, we have shown that $\Phi(U' \times W)$ is open.

Since we have shown $\Phi : U \times F \to p^{-1}(U)$ is an open continuous bijection, it is a homeomorphism. By its definition we have that the projection $U \times F \to U$ is related to $p : p^{-1}(U) \to U$ by this map. Hence the collection of such homeomorphisms (as U varies over all evenly covered neighborhoods and σ over all sheets) gives $(X \times F)/\Gamma$ the structure of a fiber bundle with fiber F.

Finally, we verify that this atlas gives $(X \times F)/\Gamma$ a Γ-structure: If two such local trivializations are defined over $b \in X/\Gamma$, then there is an evenly covered neighborhood U' of b such that the sheet maps σ, σ' giving those local trivializations restrict to two of the sheet maps for U'. Thus there exists $\gamma \in \Gamma$ such that $\sigma'\big|_{U'} = \gamma \cdot \sigma|_U$. Using the definition of the local trivialization Φ given above, it is then easy to check that the associated transition function $U' \times F \to U' \times F$ is given by

$$(u, f) \mapsto (u, \gamma \cdot f).$$

Thus we find that the transition functions of the atlas of local trivializations are given by locally constant maps to Γ, i.e. by continuous maps to Γ with the discrete topology. □

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