Math 550 – David Dumas – Spring 2019

Problem Set 5

Due Wednesday, April 3 in class

Exercises: Work these out, but do not submit them.

(E1) Let M be a manifold and G a Lie group. For maps $f,g : M \to G$, let $fg : M \to G$ denote the pointwise product map, $(fg)(x) = f(x)g(x)$. Derive a "Leibniz rule" for the Darboux derivative, expressing $(fg)^*(\omega_G)$ in terms of $f^*(\omega_G)$, f, $g^*(\omega_G)$, and g. Here ω_G is the Maurer-Cartan form.

Problems: Complete and submit two of these.

(P1) Let $\pi : P \to M$ be a principal G-bundle. A *gauge transformation of* P is a principal bundle map $\Phi : P \to P$ over the identity map of M, i.e. a smooth map $\Phi : P \to P$ such that $\Phi(u \cdot a) = \Phi(u) \cdot a$ for all $a \in G$ and $\pi(\Phi(u)) = \pi(u)$ for all $u \in P$.

(a) Since u and $\Phi(u)$ lie in the same fiber, there exists $\eta(u) \in G$ such that $\Phi(u) = u \cdot \eta(u)$. Show that the resulting map $\eta : P \to G$ is smooth and satisfies

$$\eta(u \cdot a) = a^{-1}\eta(u)a$$

for all $a \in G$. Equivalently, $\eta : P \to G^{\mathrm{inn}}$ is an equivariant map, where G^{inn} denotes the manifold G considered as a right G-space with the action $g \cdot a = a^{-1}ga$.

(b) Show that Φ is uniquely determined by function η, and that this gives a bijection between gauge transformations and sections of the associated fiber bundle $P(G^{\mathrm{inn}})$.

(P2) Let $\Phi : P \to P$ be a gauge transformation (see previous problem) with associated map $\eta : P \to G$. Let H be a connection on P (specified as a distribution), with connection form $\omega \in \Omega^1(P, \mathfrak{g})$ and curvature form $\Omega \in \Omega^2(P, \mathfrak{g})$.

(a) Show that the distribution H^Φ given by $H^\Phi(u) = d\Phi_w(H_w)$, where $w = \Phi^{-1}(u)$, is also a connection on P.

(b) Show that the connection forms ω and ω^Φ are related by $\Phi^*(\omega^\Phi) = \omega$. Equivalently, $\omega^\Phi = (\Phi^{-1})^*\omega$.

(c) Show that $\Phi^*(\omega) = \mathrm{Ad}(\eta^{-1}) \circ \omega + \eta^*(\omega_G)$.

Note: Since Φ^{-1} (the inverse diffeomorphism) is the gauge transformation corresponding to η^{-1} (the pointwise group inverse of the map η), parts (b) and (c) give a formula for ω^Φ: Simply replace η with η^{-1} in (c).

(P3) Let E be a vector bundle over M, and ∇ a connection on E. Let $\mathrm{End}(E)$ denote the vector bundle over M whose fiber over x is $\mathrm{End}(E_x)$; equivalently, $\mathrm{End}(E) = E^* \otimes E$. Thus if B is a section of $\mathrm{End}(E)$ and u is a section of E, then $B(u)$ is a section of E.

(a) Show that there is a connection ∇^{End} on $\mathrm{End}(E)$ defined by

$$\left(\nabla^{\mathrm{End}}_X B\right)(u) = \nabla_X(B(u)) - B(\nabla_X u)$$

(b) Using $\mathrm{End}(E) = E^* \otimes E$, another way to form a connection on $\mathrm{End}(E)$ is to take the tensor product of the dual connection ∇^* and ∇. Show that this also gives ∇^{End}.

(c) If A is the connection matrix of E for a given local trivialization, show that in the associated local trivialization of $\mathrm{End}(E)$ we have

$$\nabla^{\mathrm{End}} B = dB + [A, B]$$

where for a matrix-valued 1-form A and a matrix-valued function B we denote by $[A, B]$ the matrix-valued 1-form given by $[A, B](X) = A(X)B - BA(X)$.

This updated version of the assignment corrects two errors in (P2).