**Math 550 – David Dumas – Spring 2019**

# Problem Set 4

Due Monday, March 11 in class

**Problems: Complete and submit <u>two</u> of these.**

(P1) Suppose that the total space of a principal $G$-bundle admits a Riemannian metric that is invariant under the right $G$-action. Then one can define a connection (in terms of distributions) by taking $H_p$ to be the orthogonal complement of $V_p$.

Apply this construction to the round metric on $S^3$ (the one induced on the unit sphere in $\mathbb{R}^4$ by the standard inner product) and the Hopf fibration $\pi : S^3 \to \mathbb{CP}^1 \simeq S^2$, which makes $S^3$ into a principal $S^1$-bundle. Here $\pi(v)$ is the complex line spanned by $v$, where we identify $\mathbb{R}^4$ with $\mathbb{C}^2$ in the usual way, $(x_1, x_2, x_3, x_4) \mapsto (x_1 + ix_2, x_3 + ix_4)$). Express the connection as a $\mathrm{Lie}(S^1)$-valued 1-form $\omega$ written in terms of the coordinates $x_1, \ldots, x_4$ of $\mathbb{R}^4$, and then compute the associated curvature form $\Omega$.

(P2) Let $P$ and $P'$ be principal $S^1$-bundles over a manifold $M$. Then the Whitney sum $P \oplus P'$ is a fiber bundle over $M$ with typical fiber $S^1 \times S^1$ and with a $S^1$-structure. Recall that a point in $P \oplus P'$ can be identified with a pair $(p, p') \in P \times P'$ such that $\pi(p) = \pi'(p')$.

(a) Show that $S^1$ acts smoothly, freely, and properly on $P \oplus P'$ by $g \cdot (p, p') = (p \cdot g, p \cdot g^{-1})$.

(b) Let us denote the quotient of $P \oplus P'$ by the action as $P + P'$. Show that this is a principal $S^1$-bundle over $M$, and that the operation $+$ thus defined makes the set of principal $S^1$-bundles over $M$ into an abelian group.