Proposition. *Let P be a principal G-bundle over M with connection form $\omega \in \Omega^1(P,\mathfrak{g})$. Let $\Phi : P \to P$ be the map given by $\Phi(u) = u \cdot \eta(u)$. Then*

$$\Phi^*(\omega) = \mathrm{Ad}(\eta^{-1}) \circ \omega + \eta^* \omega_G \tag{1}$$

where ω_G is the Maurer-Cartan form of G.

Proof. Applying the product rule to $\Phi(u) = u \cdot \eta(u)$, for $y \in T_u P$ we have

$$d\Phi(y) = \underbrace{y \cdot \eta(u)}_{\mathrm{I}} + \underbrace{u \cdot d\eta(y)}_{\mathrm{II}}.$$

More precisely, term I refers to the image of y under the differential at u of the map $\Phi_1 : P \to P$ given by $\Phi_1(t) = t \cdot \eta(u)$, and term II refers to the image of y under the differential of the map $\Phi_2 : P \to P$ given by $\Phi_2(t) = u \cdot \eta(t)$. We calculate the terms separately.

The map Φ_1 is simply the right action on P of the fixed element $\eta(u) \in G$. Thus

$$\mathrm{I} = dR^P_{\eta(u)}(y).$$

Recall the Ad-equivariance of connection forms: $\omega(dR^P_a(y)) = dR^G_a \omega(y) = \mathrm{Ad}(a^{-1})\omega(y)$. Thus

$$\omega(\mathrm{I}) = \mathrm{Ad}(\eta(u)^{-1})(\omega(y)).$$

Next, since Φ_2 maps into a single fiber of P, the image of its differential is the infinitesimal action of some element of $\mathfrak{g}$. Specifically, if we write $\Phi_2(t) = (u \cdot \eta(u)) \cdot (\eta(u)^{-1}\eta(t))$, then the image of y by the differential of $t \mapsto \eta(u)^{-1}\eta(t)$ at $t = u$ is X_e where $X = \omega_G(d\eta(y)) \in \mathfrak{g}$ and so

$$\mathrm{II} = d\Phi_2(t) = \big(\omega_G(d\eta(y))\big)^\sharp_{u\cdot\eta(u)}.$$

Since connection forms satisfy $\omega(X^\sharp) = X$, we have

$$\omega(\mathrm{II}) = \omega_G(d\eta(y)) = \eta^*(\omega_G)(y).$$

Combining the calculations above we find

$$\begin{aligned} \Phi^*(\omega)(y) = \omega(d\Phi(y)) &= \omega(\mathrm{I}) + \omega(\mathrm{II}) \\ &= \mathrm{Ad}(\eta(u)^{-1})(\omega(y)) + \eta^*(\omega_G)(y) \end{aligned}$$

which is (1). □

Version: This is the first version of this document, released on April 4, 2019.