Math 445 – David Dumas – Spring 2019

Midterm Exam Solutions

(1) Is $\mathbb{R}$ connected in the lower limit topology?

Solution: No. We show $U = (-\infty, 0)$ and $V = [0, \infty)$ gives a separation. It is evident that U and V are disjoint and their union is $\mathbb{R}$, so we need only show they are open. Since $V = \bigcup_{n\in\mathbb{N}} [0,n)$ describes V as a union of open sets in the lower limit topology, V is open. Similarly $U = \bigcup_{n\in\mathbb{N}} [-n, 0)$ is a union of open sets, hence open.

(2) Consider these topologies on $\mathbb{Z}\times\mathbb{Z}$:
- The cofinite topology (on the set $\mathbb{Z}\times\mathbb{Z}$)
- The product of the cofinite topology on $\mathbb{Z}$ and the cofinite topology on $\mathbb{Z}$

Are these topologies the same? Is one finer than the other?

Solution: For brevity we refer to the given topologies as the cofinite topology and the product topology, respectively.

These topologies are not the same: The set $\{1\}\times\mathbb{Z}$ is closed in the product topology, but since it is infinite and not equal to $\mathbb{Z}^2$, it is not closed in the cofinite topology.

The product topology is finer. Suppose that U is open in the cofinite topology. If $U = \mathbb{Z}^2$, then it is of course also open in the product topology. Otherwise U is the complement of a finite set, $U = \mathbb{Z}^2 - \{f_1, \ldots, f_n\}$. A set that is the complement of a single point, $\mathbb{Z}^2 - \{(x,y)\}$, is open in the product topology since it is the union $((\mathbb{Z} - \{x\})\times\mathbb{Z}) \cup (\mathbb{Z}\times(\mathbb{Z}-\{y\}))$, and both sets in this union are basis elements of the product topology. Now $U = \cap_{i=1}^{n}(\mathbb{Z}^2 - \{f_i\})$, which is a finite intersection of sets that are open in the product topology, so U itself is open in the product topology as well.

(3) Let $\mathscr{T}$ denote the topology on $\mathbb{R}$ generated by the basis consisting of intervals $[a,b)$ with **rational** endpoints, i.e. where $a, b \in \mathbb{Q}$. Determine the closure of $(\sqrt{2}, \sqrt{3})$ in this topology.*

Solution: The closure is $[\sqrt{2}, \sqrt{3}]$. Since the closure of a set is the union of the set and its limit points, this is equivalent to showing that $\sqrt{2}$ and $\sqrt{3}$ are limit points of $(\sqrt{2}, \sqrt{3})$, and that no element of $\mathbb{R} - [\sqrt{2}, \sqrt{3}]$ is a limit point of $(\sqrt{2}, \sqrt{3})$.

If x is a real number that is not in $[\sqrt{2}, \sqrt{3}]$, then either $x < \sqrt{2}$ or $x > \sqrt{3}$. If $x < \sqrt{2}$, then there exist rational numbers p, q with $p \leq x < q < \sqrt{2}$. Thus $[p, q)$ is a neighborhood of x disjoint from $(\sqrt{2}, \sqrt{3})$, and x is not a limit point. Similarly, if $x > \sqrt{3}$ then there exist rational numbers p, q with $\sqrt{3} < p \leq x < q$. Then $[p, q)$ is a neighborhood of x disjoint from $(\sqrt{2}, \sqrt{3})$, hence x is not a limit point.

To show that $\sqrt{2}$ is a limit point, it suffices to show that every basis element containing $\sqrt{2}$ intersects $(\sqrt{2}, \sqrt{3})$. Let $U = [p, q)$ be such a basis element, where $p, q \in \mathbb{Q}$. Since $q > \sqrt{2}$ we have that $[p, q) \cap (\sqrt{2}, \sqrt{3})$ is nonempty.

Similarly, to show that $\sqrt{3}$ is a limit point, let $U = [p, q)$ a basis element containing $\sqrt{3}$. Since p is rational and $\sqrt{3}$ is not, we must have $p < \sqrt{3}$. But then $[p, q) \cap (\sqrt{2}, \sqrt{3})$ is nonempty.

Remark: Only in showing $\sqrt{3}$ is a limit point did we use the restriction to rational endpoints in an essential way. In the ordinary lower limit topology, $\sqrt{3}$ is *not* a limit point of $(\sqrt{2}, \sqrt{3})$, because for example $[\sqrt{3}, 100)$ is a neighborhood of $\sqrt{3}$ disjoint from $(\sqrt{2}, \sqrt{3})$. A proof similar to the one above would show that the closure of $(\sqrt{2}, \sqrt{3})$ in the ordinary lower limit topology is $[\sqrt{2}, \sqrt{3})$.

(4) Is $[0,1]^{\mathbb{N}}$ a closed set in $\mathbb{R}^{\mathbb{N}}$ with respect to the uniform topology?
(Partial credit will be given for just writing the definition of the uniform metric on $\mathbb{R}^{\mathbb{N}}$.)

Solution: The uniform metric is $\bar{\rho}(x,y) = \sup_i \min(|x_i - y_i|, 1)$.

The set $[0,1]^{\mathbb{N}}$ is closed. To show this, we must show that the complement is open. Let x be a point not in $[0,1]^{\mathbb{N}}$. Then there exists $i \in \mathbb{N}$ such that $x_i \notin [0,1]$. Since the complement of $[0,1]$ is open in the standard topology of $\mathbb{R}$, there exists $\varepsilon > 0$ so that $(x_i - \varepsilon, x_i + \varepsilon) \cap [0,1] = \emptyset$. We can also take $\varepsilon < 1$, since decreasing ε preserves this property.

We claim $B_{\bar{\rho}}(x, \varepsilon) \subset (\mathbb{R}^{\mathbb{N}} - [0,1]^{\mathbb{N}})$. If $y \in B_{\bar{\rho}}(x,\varepsilon)$ then $\bar{\rho}(x,y) < \varepsilon$, so $\min(|x_i - y_i|, 1) < \varepsilon < 1$, which gives $|x_i - y_i| < \varepsilon$, i.e. $y_i \in (x_i - \varepsilon, x_i + \varepsilon)$, and thus $y_i \notin [0,1]$. This shows $y \notin [0,1]^{\mathbb{N}}$.

We have shown that the complement of $[0,1]^{\mathbb{N}}$ contains a ball about each of its points, so this complement is open, and $[0,1]^{\mathbb{N}}$ is closed.

* There was a typographical error in this problem: The original version had the endpoints of the interval switched.