Math 445 – David Dumas – Spring 2018

Midterm Solutions

1. Consider the ordered set $X = \mathbb{N} \times \mathbb{N}$ with the dictionary order. In the order topology on X, determine whether each of the following sequences converges. If the sequence does converge, identify the point it converges to. In each case give a proof.

   (a) $a_n = n \times 2$
   (b) $b_n = 2 \times n$

   Solution.
   (a) The sequence a_n does not converge to any point of X. Suppose that we are given an open interval $U = (x \times y, z \times w)$ in X. For all $n > z$ we have $z \times w < n \times 2$ for the dictionary order on X. Therefore, a_n lies outside U for all $n \geq z$. This shows that a_n does not converge to any point in U. Since U was arbitrary, this shows a_n does not converge to any point.

   (b) The sequence b_n converges to 3×1. Let U be a neighborhood of 3×1. Then U contains an open interval containing 3×1. Since 1 is the smallest element of $\mathbb{N}$, such an open interval has the form $(x \times y, z \times w)$ with $x < 3$, and either $z > 3$ or $z = 3$ and $w > 1$. It follows that $b_n < z \times w$ for all n, and that $x \times y < b_n$ for all $n > y$. Therefore $b_n \in U$ for all $n > y$. Since U was an arbitrary neighborhood of 3×1, this shows b_n converges to 3×1.

2. (a) Write the definition of a *Hausdorff* topological space.
   (b) Prove that if X is a metrizable topological space, then X is Hausdorff.

   Solution.
   (a) A topological space X is *Hausdorff* if for all $x, y \in X$ with $x \neq y$ there exist open sets $U, V \subset X$ with $x \in U$, $y \in V$, and $U \cap V = \emptyset$.

   (b) Let X be metrizable. Let d be a metric inducing the topology of X. Let $x, y \in X$ with $x \neq y$. Let $\delta = d(x,y)/2$. Since $x \neq y$ we have $\delta > 0$. Let $U = B_d(x, \delta)$ and $V = B_d(y, \delta)$, which are open sets in X which satisfy $x \in U$ and $y \in V$. We claim $U \cap V = \emptyset$. Suppose for contradiction that $z \in U \cap V$. Then $d(x,z) < \delta$ and $d(z,y) < \delta$, and so

   $$d(x,z) + d(z,y) < 2\delta = d(x,y).$$

   This contradicts the triangle inequality satisfied by d, and hence such z does not exist. This shows $U \cap V = \emptyset$. Therefore X is Hausdorff.

3. (a) Write the definition of the *lower limit topology* on $\mathbb{R}$.
   (b) Let $A \subset \mathbb{R}$ denote the set of rational numbers in the interval $(0,1)$. Determine the closure of A in the lower limit topology on $\mathbb{R}$.

Solution.
(a) The *lower limit topology* is the topology generated by the basis

$$\mathscr{B} = \{[a,b) \mid a,b \in \mathbb{R}, a < b\},$$

that is, it is generated by the collection of all half-open intervals that contain their left endpoints.

(b) The closure is the interval $[0,1)$. To see this, first suppose that $x \in [0,1)$. Then any neighborhood U of x in the lower limit topology contains an interval $[x,y)$ for some $y > x$. Therefore U contains the open interval (x,z) where $z = \min(1,y)$. As it is an open interval in $\mathbb{R}$, (x,z) contains a rational number q. However, we also have $(x,z) \subset (0,1)$ and therefore $q \in A$. Therefore U intersects A. Since U was an arbitrary neighborhood of x, this shows x lies in the closure of A.

We have therefore shown $[0,1) \subset \bar{A}$.

Suppose that $x \notin [0,1)$. Then there exists a positive real number ε so that $U = [x, x+\varepsilon)$ satisfies $U \cap A = \emptyset$. Specifically, if $x \geq 1$ then this is true for *every* positive ε; if $x < 0$ we can take $\varepsilon = |x|/2$. Since U is a neighborhood of x in the lower limit topology, we have shown $x \notin \bar{A}$. Contrapositively, this shows $\bar{A} \subset [0,1)$.

Therefore, $\bar{A} = [0,1)$.

4. Let X be a topological space.
   (a) Write the definition of a *limit point* of a set $A \subset X$.
   (b) Suppose $A, B \subset X$ and that x is a limit point of $A \cup B$. Show that either x is a limit point of A or x is a limit point of B (or both).

Solution.
(a) An element $x \in X$ is a *limit point* of the subset $A \subset X$ if every neighborhood of x intersects $A - \{x\}$. (Equivalently, x is a limit point of A if $x \in \overline{A - \{x\}}$.)

(b) Equivalently, we must show that if x is not a limit point of A and x is not a limit point of B, then x is not a limit point of $A \cup B$. (This is the contrapositive of the given statement.)

Suppose that x is not a limit point of A and that x is not a limit point of B.

Then there exists a neighborhood U of x that is disjoint from $A - \{x\}$, and also a neighborhood V of x that is disjoint from $B - \{x\}$.

Therefore $W = U \cap V$ is a neighborhood of x. This neighborhood is disjoint from $(A \cup B) - \{x\}$. To see this, suppose on the contrary that $z \in W$ and $z \in (A \cup B) - \{x\} = (A - \{x\}) \cup (B - \{x\})$. Then either $z \in (A - \{x\})$ or $z \in (B - \{x\})$. The first possibility is a contradiction because $z \in W \subset U$ and U is disjoint from $A - \{x\}$. The second possibility is a contradiction because $z \in W \subset V$ and V is disjoint from $B -$

$\{x\}$. Therefore we obtain a contradiction in both cases and such z does not exist. We conclude that $W \cap ((A \cup B) - \{x\}) = \emptyset$, and that x is not a limit point of $A \cup B$.

5. (a) Let X be a topological space and $\sim$ an equivalence relation on X. Write the definition of the *quotient topology* on $X/\sim$.

   (b) Let $X = A \times B$ where A and B are topological spaces. Define an equivalence relation on X as follows:

$$a_1 \times b_1 \sim a_2 \times b_2 \text{ if and only if } a_1 = a_2$$

   Show that $X/\sim$ with the quotient topology is homeomorphic to A.

Solution.

(a) The quotient topology on $X/\sim$ is the unique topology so that the map $X \to X/\sim$ taking a point to its equivalence class becomes a quotient map.

Equivalently, the quotient topology on $X/\sim$ is the one in which a set is open in $X/\sim$ if and only if the union of the equivalence classes in it forms an open set in X.

(b) Such a homeomorphism can be constructed (and verified) directly, but the problem is also set up so that Corollary 22.3 can be easily applied. (This is the corollary that gives a condition for a map from X to define a homeomorphism from $X/\sim$.) Here we will use the latter approach.

Recall that if $g : X \to Z$ is a continuous surjection such that the equivalence classes of $\sim$ are exactly the preimages of points by g, then g induces a continuous bijection $f : X/\sim \to Z$, and that this map is a homeomorphism if and only if g is a quotient map.

In this case, we can take $Z = A$ and the equivalence relation $\sim$ is the one given by the surection $g : X \to A$ defined by $g(a,b) = a$. Thus g induces a continuous bijection $f : X/\sim \to A$. To show that f is a homeomorphism we need only show that g is a quotient map. However, since g is projection of $X = A \times B$ onto the first factor, it is a continuous surjection. It is also an open map. To see this, consider a basis element $U \times V$ of the product topology on X. Then $g(U \times V) = U$ is open in A. Since any open set in X is a union of basis elements, this shows that the image of any open set in X by g is also open in A, and g is an open map. Any continuous surjection that is an open map is a quotient map. Therefore g is a quotient map, and f is a homeomorphism.