Math 320 – Linear Algebra – David Dumas – Fall 2018

Solutions to 1.5.13a and 1.6.21

(1.5.13a) Let V be a vector space over a field of characteristic not equal to two. Let u and v be distinct vectors in V. Then $\{u,v\}$ is a linearly independent if and only if $\{u+v, u-v\}$ is linearly independent.

Proof. First suppose that $\{u,v\}$ is linearly independent. Suppose that we have a linear combination of $u+v$ and $u-v$ that is equal to the zero vector, i.e.

$$a(u+v)+b(u-v)=\vec{0}$$

Then expanding and regrouping terms this becomes

$$(a+b)u+(a-b)v=\vec{0}$$

By linear independence of $\{u,v\}$ we have that $a+b=0$ and $a-b=0$. Adding these equations we find $2a=0$ (where 2 denotes the element of the field that is equal to $1+1$). Since the characteristic of the field is not 2, we have $2 \neq 0$ and we can multiply by its multiplicative inverse $\frac{1}{2}$ to obtain $a=0$. Then from $a+b=0$ it follows also that $b=0$. We have therefore shown that the only linear combination $a(u+v)+b(u-v)$ that is equal to zero has $a=b=0$. Thus $\{u+v, u-v\}$ is linearly independent.

Now suppose that $\{u+v, u-v\}$ is linearly independent. Suppose that we have a linear combination of u and v that is equal to the zero vector, i.e.

$$au+bv=\vec{0}$$

Since the characteristic of the field is not 2, the field contains an element $\frac{1}{2}$ that is the multiplicative inverse of $2=1+1$ and we can write

$$u=\frac{1}{2}(u+v)+\frac{1}{2}(u-v)$$

and similarly

$$v=\frac{1}{2}(u+v)-\frac{1}{2}(u-v)$$

Substituting these into the linear combination above we find

$$a\left(\frac{1}{2}(u+v)+\frac{1}{2}(u-v)\right)+b\left(\frac{1}{2}(u+v)-\frac{1}{2}(u-v)\right)=\vec{0}$$

and after regrouping terms,

$$\frac{1}{2}(a+b)(u+v)+\frac{1}{2}(a-b)(u-v)=\vec{0}.$$

Since $\{u+v, u-v\}$ is linearly independent we conclude $\frac{1}{2}(a+b)=0$ and $\frac{1}{2}(a-b)=0$. Adding these equations we find $a=0$, and subtracting them gives $b=0$. We have therefore shown that the only linear combination of $au+bv$ that is equal to zero has $a=b=0$. Thus $\{u,v\}$ is linearly independent. $\square$

(1.6.21) Let V be a vector space over a field $\mathbb{F}$. Then V is infinite-dimensional if and only if V contains an infinite linearly independent subset.

Before giving a proof, we recall the definition: A vector space is infinite-dimensional if it does not have a finite basis.

Proof. First, suppose V contains an infinite linearly independent subset S. We will show V is infinite-dimensional.

Suppose for contradiction that V is finite-dimensional. Let $n = \dim(V)$. Since S is infinite it contains a subset S' with $n+1$ elements. Because it is a subset of a linearly independent set, S' is linearly independent. However, by Corollary 2 of Theorem 1.10, a linearly independent set in V has at most n elements, and the set S' gives a contradiction. This contradiction shows that V is infinite-dimensional.

Next, suppose V is infinite-dimensional. We will show that V contains an infinite linearly independent subset.

By Theorem 1.9, no finite set can generate V, for such a set would contain a finite basis. Since $\{\vec{0}\}$ is finite-dimensional, we know that V contains a nonzero vector. Choose one, and call it v_1. Thus $\{v_1\}$ is a linearly independent set. Now, if vectors $v_1, \ldots, v_k$ have been selected and $\{v_1, \ldots, v_k\}$ is linearly independent, we choose v_{k+1} to be a vector that is not in the span of $\{v_1, \ldots, v_k\}$. This is possible because V is not generated by the finite set $\{v_1, \ldots, v_k\}$. By Theorem 1.7, the set $\{v_1, \ldots, v_{k+1}\}$ is linearly independent.

By induction we obtain a countably infinite set of vectors $S = \{v_1, v_2, \ldots\}$ with the property that for each n, the set $\{v_1, \ldots, v_n\}$ is linearly independent. We claim that S is linearly independent.

Suppose for contradiction that S is linearly dependent, i.e. there is a linear combination of finitely many elements of S that is equal to $\vec{0}$ and in which not all coefficients are zero. Choose such a linear combination, and let n be the largest i such that v_i appears in it. Then the same expression is a linear combination of $\{v_1, \ldots, v_n\}$, hence that set is linearly dependent, a contradiction. This contradiction shows that S is linearly independent.

□

Remark 1. The most common mistake in solving this problem was to omit the argument in the last paragraph of the proof above. That is, many solutions claimed without proof that if every finite subset of S is linearly independent, then S itself is linearly independent. This is true, but some argument is needed.

Remark 2. Some students gave a solution to this problem (which is from section 1.6) using a theorem from section 1.7 of the textbook: Every vector space has a basis. This makes the problem much easier, and in this instance such solutions were not penalized. However, in general, it is expected that you solve problems in a section of the textbook without appealing to theorems proved later in the book.