Math 320 Exam 2 Solutions
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① METHOD #1.

(a) The rank of A is equal to the dimension of the span of the columns of A, which are

$$\begin{pmatrix}0\\2\\4\\-2\end{pmatrix}, \begin{pmatrix}1\\-2\\-2\\3\end{pmatrix}, \begin{pmatrix}2\\1\\6\\1\end{pmatrix}.$$

Call these a_1, a_2, a_3, respectively. Notice* $a_3 - 2a_2 = \begin{pmatrix}0\\5\\10\\-5\end{pmatrix} = 5/2\, a_1$

So these columns are not linearly independent, and their span has dimension at most 2. On the other hand, the first two columns are linearly indep since they are nonzero and neither is a multiple of the other, so the dimension is at least 2.

$$\therefore \text{Rank}(A) = \dim \operatorname{span}\left\{\begin{pmatrix}0\\2\\4\\-2\end{pmatrix}, \begin{pmatrix}1\\-2\\-2\\3\end{pmatrix}, \begin{pmatrix}2\\1\\6\\1\end{pmatrix}\right\} = 2$$

(b) Part (a) identified the first two columns as a maximal linearly indep. subset of the columns, hence they form a basis of $R(L_A)$ = span of cols.

$$\left\{\begin{pmatrix}0\\2\\4\\-2\end{pmatrix}, \begin{pmatrix}1\\-2\\-2\\3\end{pmatrix}\right\}$$ is a basis. (Note: in fact, any two columns will do.)

* How would you notice this? You see that col 1 begins with 0, while the other two do not. Then there is a unique (up to scalar multiple) combination of columns 2 and 3 which does begin with 0. That is $a_3 - 2a_2$, which is easily seen to be a multiple of a_1.
But if you don't see this, it's fine to try another method like row and column operations.

① METHOD #2.

(a) Row operations can be used to determine the rank.

$$\begin{pmatrix} 0 & 1 & 2 \\ 2 & -2 & 1 \\ 4 & -2 & 6 \\ -2 & 3 & 1 \end{pmatrix} \xrightarrow[\text{rows } 1,2]{\text{xchg}} \begin{pmatrix} 2 & -2 & 1 \\ 0 & 1 & 2 \\ 4 & -2 & 6 \\ -2 & 3 & 1 \end{pmatrix} \xrightarrow[\text{to row3}]{\text{add } -2\cdot\text{row1}} \begin{pmatrix} 2 & -2 & 1 \\ 0 & 1 & 2 \\ 0 & 2 & 4 \\ -2 & 3 & 1 \end{pmatrix} \xrightarrow[\text{to row 3}]{\text{add row 1}} \begin{pmatrix} 2 & -2 & 1 \\ 0 & 1 & 2 \\ 0 & 2 & 4 \\ 0 & 1 & 2 \end{pmatrix}$$

Now recall that the rank is equal to the maximum # of linearly independent rows, and that row + column operations do not change the rank.

In the rightmost matrix, we see rows 2,3,4 are all multiples of (0,1,2), so at most one of them can be included in a linearly indep. set, and thus the rank is at most 2.

On the other hand, rows 1 and 2 are linearly indep since they are nonzero and not proportional by a scalar, so the rank is at least 2.

$\therefore$ rank = 2. (Note: You could do more row/col ops to get $\begin{pmatrix} 1&0&0\\0&1&0\\0&0&0\\0&0&0 \end{pmatrix}$ and simplify the rest of the argument.)

(b) By (a), to find such a basis we need only find two columns of A that are linearly independent.

$\left\{ \begin{pmatrix} 0\\2\\4\\-2 \end{pmatrix}, \begin{pmatrix} 1\\-2\\-2\\3 \end{pmatrix} \right\}$ are lin. indep. since neither is a multiple of the other. Hence this is a basis of $R(L_A)$.

(2) We can compute the inverse by applying row operations to the augmented matrix.

$$(A|I) = \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \xrightarrow[1,2]{\text{xchg rows}} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \xrightarrow[\text{to row 2}]{\text{add row 1}} \left(\begin{array}{cccccc} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$$

$$\xrightarrow[\text{to row 3}]{\text{add row 1}} \left(\begin{array}{cccccc} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{array}\right) \xrightarrow[\text{to row 2}]{\text{add row 3}} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{array}\right)$$

Thus $A^{-1} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$

③ $W = \{p \mid p(0) = p(1) = 0\}$ and $Y = \{p \mid p(2) = 0\}$ and $Z = \mathrm{span}(x+1)$

(a) Yes, $P_2(\mathbb{R}) = W \oplus Y$.

Recall that the Lagrange interpolation formula gives a basis of $P_2(\mathbb{R})$, $\{p_0, p_1, p_2\}$, so that $p_0(0)=1,\ p_0(1)=0,\ p_0(2)=0$

$p_1(0)=0,\ p_1(1)=1,\ p_1(2)=0$

$p_2(0)=0,\ p_2(1)=0,\ p_2(2)=1$.

In terms of this basis, $W = \mathrm{span}(p_2)$, since any linear combination of p_0, p_1, p_2 with a nonzero coefficient for p_0 or p_1 will be nonzero at either $x=0$ or $x=1$. Similarly, $Y = \mathrm{span}(p_0, p_1)$.

Now $W + Y = P_2(\mathbb{R})$ since $W \cup Y$ contains a basis.

Also, if $p \in W \cap Y$ then writing $p = ap_0 + bp_1 + cp_2$ we see $a = b = 0$ since $p \in W$ and $c = 0$ since $p \in Y$. Thus $p = \vec{0}$, and $W \cap Y = \{\vec{0}\}$.

Since $W + Y = P_2(\mathbb{R})$ and $W \cap Y = \{\vec{0}\}$, we have $W \oplus Y = P_2(\mathbb{R})$. $\square$

(b). No, it is not true that $P_2(\mathbb{R}) = W \oplus Z$.

We saw in (a) that $\dim W = 1$. Also, $\dim Z = 1$ because it is defined as the span of a single nonzero vector.

If $P_2(\mathbb{R}) = W \oplus Z$ we would have $\dim P_2(\mathbb{R}) = \dim W + \dim Z = 2$

However $\dim P_2(\mathbb{R}) = 3$. Thus $P_2(\mathbb{R}) \neq W \oplus Z$. $\square$

(4) (a) To get the columns of $[T]_\beta^\beta$, we apply T to the basis vectors and find their coordinate vectors.

$T(1) = x \cdot 0 - 2 = -2$, so $[T(1)]_\beta = \text{column } 1 \text{ of } [T]_\beta^\beta = \begin{pmatrix} -2 \\ 0 \\ 0 \end{pmatrix}$

$T(x) = x \cdot 1 - 2x = -x$, so $[T(x)]_\beta = \text{column } 2 \text{ of } [T]_\beta^\beta = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}$

$T(x^2) = x \cdot 2x - 2x^2 = 0$, so $[T(x^2)]_\beta = \text{column } 3 \text{ of } [T]_\beta^\beta = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

Hence $[T]_\beta^\beta = \begin{pmatrix} -2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

(b) One way would be to compute the change of coordinate matrix $Q = [I]_\gamma^\beta$ and then $[T]_\gamma^\gamma = Q^{-1}[T]_\beta^\beta Q$. However we can also proceed directly by computing the γ-coordinates of the T-images of γ.

Recall $\gamma = \{x^2 + 2x + 1, 2x + 2, 2\}$.

$T(x^2+2x+1) = x(2x+2) - 2(x^2+2x+1) = -2x - 2 = 0(x^2+2x+1) - 1(2x+2) + 0(2)$

Thus $[T(x^2+2x+1)]_\gamma = \text{column } 1 \text{ of } [T]_\gamma^\gamma = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}$

$T(2x+2) = x(2) - 2(2x+2) = -2x - 4 = 0(x^2+2x+1) - 1(2x+2) - 1(2)$

Thus $[T(2x+2)]_\gamma = \text{column } 2 \text{ of } [T]_\gamma^\gamma = \begin{pmatrix} 0 \\ -1 \\ -1 \end{pmatrix}$

$T(2) = -4 = 0(x^2+2x+1) + 0(2x+2) - 2(2)$

Thus $[T(2)]_\gamma = \text{column } 3 \text{ of } [T]_\gamma^\gamma = \begin{pmatrix} 0 \\ 0 \\ -2 \end{pmatrix}$

Hence $[T]_\gamma^\gamma = \begin{pmatrix} 0 & 0 & 0 \\ -1 & -1 & 0 \\ 0 & -1 & -2 \end{pmatrix}$