Proposition 5.3.14 *let G be an algebraic semi-simple group. Let S be a surface of genus greater than 1. Then $\mathcal{D}^{vr}_{\Gamma}(S,G)$ is non-empty.*

PROOF: Choose a and b as in Lemma 5.3.13. For a pair of pants surface (in other words a sphere with three holes) one can explicitly write down a flat discrete connection such that the image of the holonomy representation contains a and b and is therefore Zariski-dense. For example one can take the flat discrete connection depicted in Figure 5.1.Gluing two pairs of pants with this discrete connection as in Figure 5.2 yields a discrete connection on a surface of genus 2, whose holonomy representation still contains a and b and thus is still very regular. If S is a surface of genus > 2 then we can decompose it along a disc into a surface of genus $g-1$ with one hole and a torus with one hole. Gluing connections as in Figure 5.2 we obtain a very regular connection on S by induction on g. Q.E.D.

5.3.4 The action is proper

We prove the second part of Theorem 5.2.6

Proposition 5.3.15 *The group G^V acts freely and properly on $\mathcal{D}^{zd}_{\Gamma}(S,G)$.*

The stabiliser of one element is identified with the centraliser of the holonomy. Hence the action is free on $\mathcal{D}^{vr}_{\Gamma}(S,G)$ which contains $\mathcal{D}^{zd}_{\Gamma}(S,G)$. The most difficult part is to verify that the G-action is proper. We cannot prove this in general here, so let us consider two special cases. The trivial case is the case where G is compact. In this case, any action of G is proper and there is nothing to show. In the non-compact case one needs again some more results on algebraic groups. An elementary proof with linear algebra methods is possible if we make two simplifications. Firstly, let us assume $\Gamma = \Gamma_g$. Then we have

$$\mathcal{M}^{vr}_{\Gamma_g}(S,G) = \mathrm{Hom}^{vr}(\pi_1(S),G)$$

and $G^V = G$. Thus the properness of the G^V-action is equivalent to the following statement:

($*$) *Let $\{\rho_n\}_{n\in\mathbb{N}} \subset \mathrm{Hom}^{vr}(A_g,G)$ be a convergent sequence. Then any sequence $\{g_n\}_{n\in\mathbb{N}} \in G$ for which $g_n\rho_n g_n^{-1}$ also converges, is bounded.*

Let $\mathcal{D} \subset \mathbb{P}(\mathbb{C}^p))^p/\mathfrak{S}^p$ be the set of p lines in $\mathbb{C}^p$ that forms a direct sum. Let us say two elements of $\mathcal{D}$ are *transerse* if no line in one element is contained in a proper subspace generated by lines of the other element.

Now let us prove the statement $(*)$ in the case of $G = \mathsf{SL}_p(\mathbb{C})$. For this we can as well assume that $\rho_n \to \bar{\rho}_0$ and that $\{g_n\}_{n\in\mathbb{N}} \in G$ satisfies $g_n \rho_n g_n^{-1} \to \bar{\rho}_1$. We need two lemmas

Lemma 5.3.16 *There exist $a, b \in \pi_1(S)$ such that $\bar{\rho}_0(a), \bar{\rho}_0(b), \bar{\rho}_1(a), \bar{\rho}_1(b)$ are all $\mathbb{C}$-split and the axis' of $\bar{\rho}_0(a)$ (respectively $\bar{\rho}_1(a)$) are transverse to the axis' of $\bar{\rho}_0(b)$ (respectively $\bar{\rho}_1(b)$).*

PROOF: We distinguish two cases depending on whether $\bar{\rho}_0$ and $\bar{\rho}_1$ are conjugate or not. In the former case it suffices to choose a and b in such a way that $\bar{\rho}_0(a)$ and $\bar{\rho}_0(b)$ are $\mathbb{C}$-split with distinct axis. Now the set of $\mathbb{C}$-split elements of G is Zariski-open and non-empty and thus contains an element $\bar{\rho}_0(a)$ from the Zariski-dense subset $\bar{\rho}_0(A_g)$. Moreover, the set of $\mathbb{C}$-split elements of G with axis' transverse to a given element of $\mathcal{D}$ is still Zariski open and this allows us to chose b. This settles the case where $\bar{\rho}_0$ and $\bar{\rho}_1$ are conjugate. In the other case it follows from Proposition 5.3.6 that $(\rho_0 \times \rho_1)(\pi_1(S))$ is Zariski dense in $G \times G$. Then the argument is reduced to the first case. Q.E.D.

Let us choose a and b as in the lemma. As $\rho_n(a) \to \bar{\rho}_0(a)$ and the set of $\mathbb{C}$-split elements in G is open, we may assume that $\rho_n(a)$ is $\mathbb{C}$-split for all $n \in \mathbb{N}$.

Let $\alpha_n \in \mathcal{D}$ denote the set of axis' of $\rho_n(a)$. Then $\alpha_n \to \bar{\alpha}_0$, where $\bar{\alpha}_0$ is the set of axis' of $\bar{\rho}_0(a)$. Similarly, $g_n\alpha_n \to \bar{\alpha}_1$, where $\bar{\alpha}_1$ is the set of axis' of $\bar{\rho}_1(a)$.

We now remark that we have a locally bounded map W from $\mathcal{D}$ to $\mathsf{SL}_p(\mathbb{C})$, so that for any $\beta \in \mathcal{D}$, $W(\beta)\cdot\beta = \bar{\alpha}_1$: we choose a metric on $\mathbb{C}^p$ and for any $\beta \in \mathcal{D}$, $W(\beta)$ to send unit vectors in the lines of β to unit vectors in the lines of $\bar{\alpha}_1$. It follows that we can write

$$g_n = v_n^{-1} t_n u_n,$$

where t_n is diagonal with respect to a basis of eigenvectors of $\bar{\alpha}_1$, $u_n = W(\alpha_n)$, and $v_n = W(g_n\alpha_n)$. Since W is a locally bounded map, we can reduce our discussion to the case where g_n are diagonal matrices with respect to a basis of eigenvectors of $\bar{\alpha}_1 = \bar{\alpha}_0$.

For that we need the following lemma:

Lemma 5.3.17 *Let $\{g_n\}_{n\in\mathbb{N}}$ be an unbounded sequence of diagonal matrices in $\mathsf{SL}_p(\mathbb{C})$. Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence of elements of $\mathbb{P}(\mathbb{C}^p)$ which converges to a line, which is not included in a coordinate hyperplane. Then $\{g_n x_n\}_{n\in\mathbb{N}}$ converges (after taking subsequences) to a line contained in a coordinate hyperplane.*

PROOF: Let first give the proof for $p = 2$. We then have $g_n = \mathrm{diag}(\lambda_n, \lambda_n^{-1})$ and we may assume that $\lambda_n \to \infty$. If $x_n = [a_n : b_n]$ converges to $[a : b]$ with $a, b \neq 0$. Then

$$g_n x_n = [\lambda_n a_n : \lambda_n^{-1} b_n] \to [1 : 0].$$

For the general case, we may as well assume – after possibly permuting the axis' and choosing subsequences – that if $g_n = \mathrm{diag}(\lambda_n^1, \ldots, \lambda_n^p)$, then for some q, with $1 < q \leq p$,

- $\{\lambda_n^k\}_{n\in\mathbb{N}}$ is bounded for $k \geq q$,
- $\{\lambda_n^k\}_{n\in\mathbb{N}}$ tends to ∞ is bounded for $k < q$,

Then it follows that, after taking subsequences, the limit of $\{g_n x_n\}_{n\in\mathbb{N}}$ lies in the hyperplane on which the last coordinate vanishes. Q.E.D.

Now we can finish the proof: we now use our hypothesis on b. Let $\beta_n, \bar\beta_0$ and $\bar\beta_1$ denote the set of axis' of $\rho_n(b), \bar\rho_0(b)$ and $\bar\rho_1(b)$ respectively. Then $g_n\beta_n \to \bar\beta_1$ and $\beta_n \to \bar\beta_0$. Write all elements of G as matrices with respect to a basis of eigenvectors of $\bar\alpha_1 = \bar\alpha_0$. Then the matrices $g_n \in \mathsf{SL}_p(\mathbb{C})$ are diagonal. Also as $\bar\beta_0$ is transverse to $\bar\alpha_0 = \bar\alpha_1$, each line in β_n converges to an axis', which is not included in a coordinate hyperplane. Thus, by the lemma, if g_n were unbounded then $g_n\beta_n$ would converge to an element in $\mathcal{D}$ not transverse to $\bar\alpha_1$. But this contradicts the choice of a and b. Therefore $\{g_n\}_{n\in\mathbb{N}}$ must be bounded. This finishes our indication of the proof of (iii).

Hint of the proof in the general case. We just sketch here how the general proof works. First we need the following fact about actions of algebraic groups: to show that an algebraic action of an algebraic group on an algebraic variety is proper, it suffices to show that the orbits are closed. Indeed, by Rosenlicht theorem [Ros63] the action is stratified in the following sense: there is a sequence of Zariski closed sets

$$V_n \subset V_{n-1} \subset V_{n-2} \subset \ldots \subset V_1 \subset V_0 = V,$$

such that the action on $V_i \setminus V_{i+1}$ is proper.

In our case, the fact that the orbit are closed follows from Corollary 5.3.7: by this corollary, the G-orbit of a representation ρ, is

$$V_\rho = \{\overline{\rho} \mid \forall\gamma \in \pi_1(S)\ \ \mathrm{Tr}(\rho(\gamma)) = \mathrm{Tr}(\overline{\rho}(\gamma))\}.$$