Math 549: Differentiable Manifolds I – David Dumas – Fall 2017

Topic Outline

This is an outline of the topics we have covered this semester. It is in roughly chronological order.

(1) First definitions
  (a) Locally Euclidean topological space
  (b) Topological n-manifold
  (c) Smooth (C^∞) function on open set in $\mathbb{R}^n$
  (d) Diffeomorphism between open sets in $\mathbb{R}^n$
  (e) Smooth atlas, smooth structure on a topological manifold
  (f) Smooth manifold, smooth manifold with boundary
  (g) Smooth (C^∞) function on a smooth manifold
  (h) Smooth map between smooth manifolds
  (i) Diffeomorphism of smooth manifolds
  (j) Open submanifold

(2) First examples
  (a) Vector spaces over $\mathbb{R}$ and $\mathbb{C}$
  (b) The unit sphere $S^n \subset \mathbb{R}^{n+1}$
  (c) The n-torus $T^n = S^1 \times \cdots \times S^1$
  (d) Real projective space $\mathbb{RP}^n$
  (e) Complex projective space $\mathbb{CP}^n$
  (f) The Grassmannian $G_k(V)$ of k-dimensional subspaces of a finite-dimensional $\mathbb{R}$-vector space V
  (g) The upper half space (a manifold with boundary)
  (h) The closed ball (a manifold with boundary)

(3) Bump functions and partitions of unity
  (a) Existence of nontrivial compactly supported smooth functions on $\mathbb{R}$
  (b) Bump functions on $\mathbb{R}$ (equal to 1 on an interval, compactly supported)
  (c) Bump functions on $\mathbb{R}^n$ and in smooth manifolds (equal to 1 on a neighborhood of a point)
  (d) Compact exhaustions of topological manifolds and existence of locally finite coverings
  (e) Partition of unity on a manifold: Definition, existence (subordinate to any cover)

(4) Tangent vectors and differentials
  (a) Definition of a tangent vector at p as a derivation of $C^\infty(M)$ at p
  (b) Space T_pM is isomorphic to $\mathbb{R}^{\dim M}$
  (c) Local coordinates $x_1, \ldots, x_n$ give basis $\left.\frac{\partial}{\partial x_1}\right|_p, \ldots \left.\frac{\partial}{\partial x_n}\right|_p$
  (d) Locality: Isomorphism $T_pU \simeq T_p\mathbb{R}^n$ for $U \subset \mathbb{R}^n$ open
  (e) Tangent space to open submanifold
  (f) Tangent vectors as velocity vectors of curves
  (g) Differential of a smooth map: abstractly, and in local coordinates

(5) Special classes of smooth maps
  (a) Definitions: immersion, submersion, smooth embedding, smooth covering, local diffeomorphism
  (b) Immersion and submersion are local properties, while embedding and covering are not
  (c) Injective smooth immersion need not be a smooth embedding (e.g. lemniscate, irrational line on torus)

(d) The Inverse Function Theorem (IFT)
(e) IFT $\implies$ An immersion is locally modeled on the inclusion of R^k into R^n for some $k \leq n$. (Local standard form of an immersion)
(f) IFT $\implies$ A submersion is locally modeled on the projection of R^k onto R^n for some $k \geq n$. (Local standard form of a submersion)
(g) The *Rank Theorem* is the generalization of these local models to maps of constant rank.
(h) Theorem: If $F : M \to N$ is a submersion, then $F^{-1}(q)$ is an embedded submanifold of M of dimension $\dim(M) - \dim(N)$. Idea: Local standard form gives charts for $F^{-1}(q)$.
(i) Immersed and embedded submanifolds
(j) Slice charts for embedded submanifolds

(6) Regular and critical values of smooth maps
(a) Definition of critical point, regular value and critical value of a smooth map.
(b) Theorem: If q is a regular value of a smooth map $F : M \to N$, then $F^{-1}(q)$ is an embedded submanifold of M.
(c) Definition of a set of measure zero in $\mathbb{R}^n$
(d) Definition of a set of measure zero in a smooth manifold
(e) Sard's Theorem: The set of critical values of a smooth map has measure zero. (We skipped all discussion of the proof.)
(f) Corollary: Every smooth map has a regular value
(g) Corollary: A manifold of dimension k cannot surject a manifold of dimension n by a smooth map if $k < n$.

(7) Tangent bundle and vector fields
(a) The tangent bundle as a union of tangent spaces
(b) Smooth structure on the tangent bundle, making it a manifold of dimension $2\dim(M)$
(c) The projection $\pi : TM \to M$ is a submersion
(d) Definition of parallelizability
(e) $\mathbb{R}^n$ is parallelizable
(f) Vector fields as sections of TM
(g) Vector fields as derivations of $C^\infty(M)$
(h) Vector field X in local coordinates: $X = \sum_i a_i \frac{\partial}{\partial x_i}$. Smoothness is equivalent to $a_i \in C^\infty(M)$.
(i) $\mathrm{Vect}(M)$, the set of all smooth vector fields, is a module over $C^\infty(M)$.
(j) Definition of the pushforward of a vector field by a diffeomorphism.

(8) Integral curves and flows
(a) Definition of an integral curve of a vector field
(b) Being an integral curve is an ODE
(c) A smooth vector field has a unique maximal integral curve through a given point in the manifold. Proof using fundamental existence/uniqueness theorem in ODE.
(d) The (partial) flow of a vector field, $\Theta : \mathscr{D} \to M$ where $\mathscr{D} \subset \mathbb{R} \times M$ is an open neighborhood of $\{0\} \times M$.
(e) X is *complete* if $\mathscr{D} = \mathbb{R} \times M$
(f) Complete vector field induces a flow $\theta : \mathbb{R} \times M \to M$, $(t,m) \mapsto \theta_t(m)$, satisfying $\theta_t \circ \theta_s = \theta_{s+t}$.
(g) Conversely, a flow is uniquely determined by its generating vector field X, where $X_m = \frac{\partial}{\partial t}\theta_t(m)\Big|_{t=0}$
(h) Every vector field on a compact manifold is complete

(9) Lie groups and actions
   (a) Definition of a Lie group
   (b) Examples: Vector spaces over $\mathbb{R}$ and $\mathbb{C}$, S^1, $\mathbb{R}^*$, $\mathbb{C}^*$, $\mathrm{GL}_n(\mathbb{R})$, $\mathrm{SL}_n(\mathbb{R})$, $\mathrm{SO}(n)$, $\mathrm{SU}(n)$
   (c) Left and right translation diffeomorphisms of a Lie group
   (d) Lie groups are parallelizable
   (e) Definition of Lie group homomorphism, isomorphism
   (f) Lie group homomorphisms are maps of constant rank
   (g) Lie group homomorphism is an isomorphism if and only if it is injective
   (h) Lie subgroups (reminder: are *immersed* submanifolds)
   (i) Definition of an action of a Lie group on a smooth manifold
   (j) Examples of actions ($\mathrm{Aut}(V)$ acts on V, $\mathrm{GL}_n(\mathbb{R})$ acts on $\mathbb{R}^n$, G acts on G, $G \times G$ acts on G, $\ldots$)
   (k) A flow is equivalently an action of $\mathbb{R}$
   (l) Definition of stabilizer, orbit under Lie group action

(10) Lie algebras, Lie derivative, Lie bracket
   (a) Definition of a Lie algebra (over $\mathbb{R}$)
   (b) Examples: $(\mathbb{R}^3, \times)$, commutator in an associative $\mathbb{R}$-algebra
   (c) Lie bracket $[X,Y]$ of vector fields as commutator of derivations
   (d) $(\mathrm{Vect}(M), [\cdot,\cdot])$ is a Lie algebra
   (e) Lie derivative $L_X Y$ as a derivative of Y along the flow of X
   (f) Theorem: $[X,Y] = L_X Y$
   (g) Lie bracket in local coordinates
   (h) The Lie algebra $\mathrm{Lie}(G)$ of a Lie group G: The space of left-invariant vector fields on G
   (i) $\mathrm{Lie}(G)$ is a Lie subalgebra of $\mathrm{Vect}(G)$
   (j) $\mathrm{Lie}(G) \simeq T_e G$ (an isomorphism of $\mathbb{R}$-vector spaces)
   (k) The differential at e of a Lie group homomorphism is a Lie algebra homomorphism
   (l) Left-invariant vector fields are complete
   (m) Definition of the exponential map $\exp : \mathrm{Lie}(G) \to G$
   (n) The image of $\mathbb{R} \cdot x \subset \mathrm{Lie}(G)$ by the exponential map is a 1-parameter subgroup of G

(11) Cotangent spaces, cotangent vectors, and 1-forms
   (a) The cotangent space $T_p^* M$
   (b) Local coordinates x_i give a basis $dx_i|_p$ of $T_p^* M$
   (c) Cotangent bundle $T^* M$ as the union of cotangent spaces
   (d) A 1-form is a section of the cotangent bundle
   (e) Local expression for a 1-form: $\eta = \sum_i a_i dx_i$, $a_i \in C^\infty(M)$.
   (f) Equivalent conditions for smoothness of a 1-form
      (i) Smooth coordinate functions
      (ii) Smooth as a map to $T^* M$ (for a certain smooth structure on $T^* M$)
      (iii) Gives a smooth function when evaluated on a vector field
   (g) Definition of the pullback of a 1-form by a smooth map
   (h) Relation of pullback F^* to adjoint of the linear map dF
   (i) The differential df of a function $f \in C^\infty(M)$ is a 1-form
   (j) The space $\Omega^1(M)$ of 1-forms is a $C^\infty(M)$ module
   (k) Integral of a 1-form on $\mathbb{R}$ over a closed interval $[a,b]$
   (l) Integral of a 1-form on M over an oriented curve

(12) Riemannian manifolds
   (a) Definition of Riemannian metric, Riemannian manifold

(b) Examples: $\mathbb{R}^n$, induced metric on a submanifold of $\mathbb{R}^n$, 2-sphere with round metric, hyperbolic plane $\mathbb{H}^2$
(c) Writing a Riemannian metric in local coordinates: $g = \sum_{i,j} g_{ij} dx_i \otimes dx_j$, $g_{ij} \in C^\infty(M)$
(d) Symmetric tensor product, $\alpha\beta = \alpha \odot \beta = \frac{1}{2}(\alpha \otimes \beta + \beta \otimes \alpha)$
(e) Writing a Riemannian metric using the symmetric tensor product (e.g. $dx^2 + dy^2 - dxdy$)
(f) Using a Riemannian metric g to define:
  (i) Length of a tangent vector
  (ii) Angle between two tangent vectors
  (iii) Length of a curve
  (iv) Distance between two points (infimum of path length)
  (v) Ball of radius r centered at a point
(g) Riemannian distance function gives M the structure of a metric space
(h) Every manifold has a Riemannian metric
(i) Isomorphism $T_pM \simeq T_p^*M$ induced by a Riemannian metric
(j) "Musical" isomorphisms $\mathrm{Vect}(M) \leftrightarrow \Omega^1(M)$ induced by a Riemannian metric

(13) Exterior algebra and differential forms
(a) $\bigotimes^k V^*$ is the space of multilinear maps $V \times \cdots \times V \to \mathbb{R}$.
(b) $\bigwedge^k V^*$ is the space of alternating multilinear maps $V \times \cdots \times V \to \mathbb{R}$.
(c) $\Sigma^k V^*$ is the space of symmetric multilinear maps $V \times \cdots \times V \to \mathbb{R}$, but these play less of a role in this course
(d) Definition of $\alpha_1 \wedge \alpha_2 \wedge \cdots \wedge \alpha_n \in \bigwedge^k V^*$, for $\alpha_i \in V^*$; value on vectors $v_1, \ldots, v_k$ is the determinant of $(\alpha_i(v_j))_{ij}$
(e) For example, $\alpha \wedge \beta = \alpha \otimes \beta - \beta \otimes \alpha$ if $\alpha, \beta \in V^*$
(f) Basis for $\bigwedge^k V^*$ in terms of a given basis of V^*; multi-index notation ε_I
(g) Wedge product in general: $\wedge : \left(\bigwedge^k V^*\right) \times \left(\bigwedge^\ell V^*\right) \to \bigwedge^{k+\ell} V^*$
(h) $\bigwedge^\bullet V^* = \bigoplus_k \bigwedge^k V^*$, the exterior algebra of V^*
(i) Associativity and graded-commutativity of $\wedge$
(j) Definition of $\Omega^k(M)$, the space of smooth k-forms
(k) $\Omega^k(M)$ is a module over $C^\infty(M)$
(l) Equivalent conditions for the smoothness of a k-form (coordinates are smooth, section map is smooth, action on a tuple of smooth vector fields is smooth)

(14) Orientations
(a) (We only consider finite-dimensional $\mathbb{R}$-vector spaces here)
(b) Orientation of a $\mathbb{R}$-vector space V as an equivalence class of ordered bases
(c) Orientation of a $\mathbb{R}$-vector space V as a connected component of $\bigwedge^{\dim V} V^*$
(d) Equivalence of these definitions
(e) Every vector space has exactly two orientations
(f) The standard orientation of $\mathbb{R}^n$ (the vector space)
(g) Pointwise orientation of a manifold
(h) Orientation of a manifold as a smoothly varying pointwise orientation
(i) The standard orientation of $\mathbb{R}^n$ (the manifold)
(j) Orientation of a manifold is equivalent to a nowhere-zero element of $\Omega^n(M)$
(k) Definition of orientability
(l) A connected manifold has either zero or two orientations
(m) Lie groups are orientable

(15) Integration
  (a) The space $\Omega_c^k(M)$ of compactly-supported k-forms
  (b) Integral of a compactly supported n-form on $\mathbb{R}^n$
  (c) Definition of the integral $\int_M \omega$ where $\omega \in \Omega_c^n(M)$ and M is an oriented manifold (using a partition of unity)
  (d) Properties of the integral
    (i) Independent of the partition of unity
    (ii) Linearity
    (iii) Invariance under orientation-preserving differomorphism
    (iv) Positive for orientation forms
    (v) Change of sign under change of orientation
  (e) Volume form ω_g on an oriented Riemannian manifold (M,g)
  (f) Volume of a compact oriented Riemannian manifold
  (g) Integral of a compactly supported function over an oriented Riemannian manifold

(16) Exterior derivative, Lie derivative, and interior products of forms
  (a) Theorem: There exists a unique collection of maps $d : \Omega^k(M) \to \Omega^{k+1}(M)$ satisfying
    (i) $\mathbb{R}$-linearity
    (ii) $d(\alpha \wedge \beta) = (d\alpha) \wedge \beta + (-1)^k \alpha \wedge (d\beta)$, where $\alpha \in \Omega^k(M)$
    (iii) $d \circ d = 0$
    (iv) $(df)(X) = Xf$ for $f \in C^\infty(M) = \Omega^0(M)$
  (b) Determining a coordinate formula for d using these properties
  (c) Coordinate-free expression for d using Lie bracket
  (d) Lie derivative $L_V \omega$ for $V \in \mathrm{Vect}(M)$, $\omega \in \Omega^\bullet(M)$
  (e) Expression for $L_V \omega(X_1, \ldots, X_k)$ in terms of the Lie bracket of vector fields
  (f) Interior product $i_V : \Omega^k(M) \to \Omega^{k-1}(M)$
  (g) Cartan's "magic" formula $L_V \omega = i_V(d\omega) + d(i_V \omega)$

(17) (Cartan-)Stokes Theorem
  (a) Boundary orientation of an oriented manifold with boundary
  (b) Theorem: If $\omega \in \Omega_c^{n-1}(M)$, M a smooth n-manifold with boundary, then

$$\int_M d\omega = \int_{\partial M} \omega.$$

  (c) Proof: Reduce to case of a form supported in a chart; use linearity, Fubini, and fundamental theorem of calculus.
  (d) Special case: Integral of $d\omega$ over a compact manifold (without boundary) is zero
  (e) Special case: Integral of df over a 1-manifold with boundary is a sum of $\pm$values of f at boundary points
  (f) Special case: All of the classical vector calculus integral theorems in $\mathbb{R}^2$, $\mathbb{R}^3$
  (g) Divergence of a vector field on an oriented Riemannian manifold, $\operatorname{div} X \in C^\infty(M)$, uniquely determined by $(\operatorname{div} X)\omega_g = d(i_X \omega_g)$
  (h) Divergence Theorem on a Riemannian manifold, reduction to Cartan-Stokes Theorem

(18) de Rham cohomology
  (a) Definitions: Closed form, exact form, cohomologous forms
  (b) $H^k(M) = \{$ closed k-forms $\}/\{$ exact k-forms $\}$ is a $\mathbb{R}$-vector space
  (c) $H^\bullet(M) = \bigoplus_k H^k(M)$ is a graded-commutative algebra over $\mathbb{R}$; product induced by wedge of forms
  (d) $[\omega]$ denotes the class of a closed k-form ω in $H^k(M)$

(e) If $F : M \to N$ is smooth, then pullback of forms induces an $\mathbb{R}$-algebra homomorphism $F^* : H^\bullet(N) \to H^\bullet(M)$.
(f) $H^\bullet(M \coprod M') \simeq H^\bullet(M) \oplus H^\bullet(M')$
(g) Diffeomorphic manifolds have isomorphic cohomology algebras
(h) Definition of smooth homotopy of maps, smooth homotopy equivalence of manifolds
(i) Theorem: Smoothly homotopic maps induce the same pullback map on cohomology
(j) Proof using homotopy operator $h : \Omega^k(M \times I) \to \Omega^{k-1}(M)$
(k) Corollary: Smoothly homotopy equivalent manifolds have isomorphic cohomology

(19) Calculating de Rham cohomology
  (a) $H^0(M)$ is the set of locally constant functions on M, hence $H^0(M) \simeq \mathbb{R}^E$ where E is the set of connected components of M
  (b) Direct calculation of $H^\bullet(S^1)$
  (c) Mayer-Vietoris sequence
  (d) Application: Inductive calculation of $H^\bullet(S^n)$
  (e) Application: de Rham cohomology of punctured $\mathbb{R}^n$
  (f) For M compact, connected, and oriented (no boundary):
    (i) $H^n(M) \simeq \mathbb{R}$
    (ii) Integration provides such an isomorphism, $[\alpha] \mapsto \int_M \alpha$
    (iii) Cartan-Stokes theorem shows the integration map is well-defined.
    (iv) In particular $[\omega]$ is nonzero if ω is a volume form (and same for $[f\omega]$ if $f \geq 0$ and f not identically zero)
    (v) Showing that the integration map has no kernel uses the Poincareé Lemma with Compact Support (Lee, Lemma 17.27)

(20) Degree theory
  (a) Equivalent definitions of the degree $\deg(F)$ for $F : M \to N$, with M and N compact, connected, oriented manifolds of the same dimension:
    (i) For $\omega \in \Omega^n(N)$ we have

$$\int_M F^*\omega = \deg(F) \int_N \omega$$

    (ii) For a regular value q of F, we have

$$\deg(F) = \sum_{p \in F^{-1}(q)} \operatorname{sgn}(p)$$

    where $\operatorname{sgn}(p) = 1$ if dF_p is orientation-preserving, and $\operatorname{sgn}(p) = -1$ if dF_p is orientation-reversing.
  (b) Using integration isomorphisms $H^n(M) \simeq \mathbb{R}$ and $H^n(N) \simeq \mathbb{R}$, the pullback map F^* on H^n corresponds to multiplication by $\deg(F)$ (equivalent to definition (1) above)
  (c) Homotopic maps have the same degree
  (d) $\deg(G \circ F) = \deg(G)\deg(F)$
  (e) Degree of a diffeomorphism is ± 1 depending on whether it preserves orientation
  (f) Degree of the antipodal map of S^n is $(-1)^{n+1}$
  (g) For self-maps of S^n, any map without fixed points is homotopic to the antipodal map.
  (h) Application: Smooth Brouwer theorem: Every smooth map from the closed n-ball to itself has a fixed point
  (i) Application: S^n has a nowhere-vanishing vector field if and only if n is odd

(21) Poincaré Duality
  (a) Statement of Poincaré duality for compact oriented n-manifold

(i) As an isomorphism $H^k(M) \simeq \left(H^{n-k}(M)\right)^*$
(ii) As a nondegenerate bilinear pairing $H^k(M) \times H_c^{n-k}(M) \to \mathbb{R}$, given by

$$([\alpha],[\beta]) \mapsto \int_M \alpha \wedge \beta$$

(b) Compactly supported de Rham cohomology $H_c^\bullet(M)$
(c) Statement of Poincaré duality for oriented n-manifold, as duality between $H^\bullet$ and $H_c^\bullet$
(d) Calculating $H_c^\bullet(\mathbb{R}^n)$
(e) Sketch of proof of Poincaré duality theorem for compact manifolds
  (i) The Five Lemma
  (ii) Extension by zero maps and the Mayer-Vietoris sequence for $H_c^\bullet$
  (iii) Good covers: Definition, existence
  (iv) A compact manifold has a finite good cover
  (v) Inductive proof of PD for any oriented manifold with finite good cover:
    (A) Base case: $H^\bullet(\mathbb{R}^n)$ and $H_c^\bullet(\mathbb{R}^n)$.
    (B) Inductive step: Write M as $U \cup V$, where $U = (U_1 \cup \cdots \cup U_N)$ and $V = U_{N+1}$. Apply the Five Lemma to Mayer-Vietoris for $H^\bullet$ and the dual of Mayer-Vietoris for $H_c^\bullet$.

(22) Distributions and the Frobenius Theorem
(a) Definition of a smooth distribution $D \subset TM$ of rank k (smoothness using local frames)
(b) The space of sections $\Gamma(D)$ and local sections $\Gamma(U,D)$, $U \subset M$
(c) Definition of an integral manifold of a distribution: Immersed submanifold $N \hookrightarrow M$ such that $T_pN = D_p$ for all $p \in N$.
(d) *Integrable* distribution: There exists an integral submanifold through each point
(e) The span of a nowhere-vanishing vector field is an integrable distribution of rank 1
(f) The distribution of rank 2 on $\mathbb{R}^3$ spanned by $X = \frac{\partial}{\partial x} + y\frac{\partial}{\partial z}$ and $Y = \frac{\partial}{\partial y}$ is not integrable
(g) *Involutive* distibution: $\Gamma(D)$ is a Lie subalgebra of $\mathrm{Vect}(M)$.
(h) Annihilator $\mathscr{I}(D) \subset \Omega^\bullet(M)$ of a distribution D is an ideal
(i) *Strongly integrable* distribution: Everywhere locally expressible as span of $\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_k}$ for a suitable local coordinate system $(x_1, \ldots, x_n)$.
(j) Frobenius theorem: The following are equivalent:
  (i) D is strongly integrable
  (ii) D is integrable
  (iii) D is involutive
  (iv) $\mathscr{I}(D)$ is a differential ideal
(k) Proof sketch:
  (i) Strongly integrable $\implies$ integrable: immediate
  (ii) Integrable $\implies$ involutive: The Lie bracket can be computed on a submanifold
  (iii) Involutive $\implies$ strongly integrable: This is the core of the proof; the key is to construct a Lie-commuting local frame for D, whose commuting flows then give the adapted local coordinates
  (iv) Involutive $\iff$ $\mathscr{I}(D)$ is a differential ideal: Not difficult, but we skipped this