Math 549: Differentiable Manifolds I – David Dumas – Fall 2017

Midterm Exam

Due Monday, October 16 at 9:00am in Taft 305

No collaboration is allowed on the exam. You may consult your notes and the course textbooks (as listed on the syllabus) when working on these problems, but seeking assistance from other resources is not permitted.

In general, you need to prove every claim you make in your solutions. You may cite theorems from lecture or from the sections of the main textbook (Lee) that have been covered thus far. Ask the instructor before citing theorems from the other textbooks or from sections of the text not yet discussed in lecture. Also ask if you are uncertain about how much detail or justification is required.

(1) Is the open unit ball $\{(x,y,z) \mid x^2+y^2+z^2<1\}$ diffeomorphic to the open unit cube $(-1,1)^3$? Either construct an explicit diffeomorphism (with proof), or show that these open subsets of $\mathbb{R}^n$ are not diffeomorphic.

(2) Consider the trace function $\mathrm{tr} : \mathrm{SL}(2,\mathbb{R}) \to \mathbb{R}$. What are the regular values of this smooth map?

(3) Define $\tilde{F} : \mathbb{S}^2 \to \mathbb{R}^4$ by $F(x,y,z) = (x^2-z^2, xy, xz, yz)$. Note that $\tilde{F}(-x,-y,-z) = \tilde{F}(x,y,z)$, and that $\mathbb{RP}^2 = \mathbb{S}^2/(p \sim -p)$, therefore $\tilde{F}$ induces a map $F : \mathbb{RP}^2 \to \mathbb{R}^4$. Show that F is a smooth embedding.

(4) The Lie group $G = \mathrm{GL}(n,\mathbb{R})$ is an open submanifold of the vector space $M_{n\times n}$ of $n \times n$ real matrices, hence the tangent space T_eG is naturally identified with $M_{n\times n}$. Let $v \in M_{n\times n} \simeq T_eG$, and let $\tilde{v} \in \mathrm{Lie}(G)$ denote its extension to a left-invariant vector field on G. For $i,j \in \{1,\ldots,n\}$, let $f_{ij} \in C^\infty(G)$ denote the function which maps a matrix to its i,j entry. Compute the smooth function $\tilde{v}f_{ij} \in C^\infty(G)$ by giving an explicit formula for $(\tilde{v}f_{ij})(g)$ in terms of the matrix entries $g_{k\ell}$ of g. Note that Xf denotes the action of a vector field X on a smooth function f as a derivation.

(5) Let $\gamma : S^1 \to M$ be a smooth embedding. Show that there exists a smooth vector field X on M that does not vanish on $\gamma(S^1)$ and which is tangent to $\gamma(S^1)$ at each point. (The last condition means that for each point $p = \gamma(t)$, the vector X_p is a multiple of $\gamma'(t)$.)

(6) Let V be a finite-dimensional vector space over $\mathbb{R}$ and $W \subset V$ a subspace. Let $H \subset \mathrm{GL}(V)$ denote the set of invertible linear transformations $T : V \to V$ such that $T(W) \subset W$.
   (a) Show that H is an embedded Lie subgroup of $\mathrm{GL}(V)$.
   (b) Let $r : H \to \mathrm{GL}(W)$ denote the map defined by $r(T) = T|_W$. That is, $r(T)$ is the linear transformation $W \to W$ obtained by restricting T. Show that r is a Lie group homomorphism and a submersion.
   (c) Let $s : H \to \mathrm{GL}(V/W)$ denote the map defined by $s(T)(v+W) = T(v)+W$. Show that s is a Lie group homomorphism and a submersion.