Math 549: Differentiable Manifolds I – David Dumas – Fall 2017

Homework 9

Due Monday, November 13 at 1:00pm

(1) Let M be a smooth manifold. Show that the tangent bundle TM is orientable. [LEE]

(2) Show that the form $\eta \in \Omega^2(\mathbb{S}^2)$ defined by $\eta_p(v,w) = \det(p,v,w)$ is equal to the Riemannian area form for the round metric (that is, the Riemannian metric which is simply the restriction of $g_{\mathbb{R}^3}$.) [ND]

(3) Calculate the Riemannian area form of the hyperbolic plane. Write it as $f\,dx \wedge dy$ for an explicit function f.

(4) In a Riemannian manifold (M,g), the sphere of radius r centered at a point p is the set

$$S(p,r) = \{q \in M \mid d(p,q) = r\}$$

and the closed ball of radius r is

$$\bar{B}(p,r) = \{q \in M \mid d(p,q) \leq r\}.$$

When $\dim M = 2$, the set $S(p,r)$ is also called a circle of radius r.

For a point p in the hyperbolic plane $\mathbb{H}^2$, calculate the area of the ball $\bar{B}(p,r)$ and the length of the circle $S(p,r)$, each as an explicit function of r.

You may use the following fact:

Proposition. *A Riemannian ball in the hyperbolic plane, when considered as a subset of $\mathbb{R}^2$, is also a Euclidean ball (though its Euclidean center is not the same as its hyperbolic center).*

Hint: Problem 1 on Homework 8 implies the answer does not depend on p. (Explain why.) Problem 5 on Homework 7 and the proposition above should allow you to determine the Riemannian circle/ball explicitly in order to calculate the necessary integrals.

(5) Lee, Chapter 16, Problem 2, on page 434.

(6) Lee, Chapter 16, Problem 10, on page 435.