Math 549: Differentiable Manifolds I – David Dumas – Fall 2017

Homework 8

Due Monday, November 6 at 1:00pm

(1) Let (M,g) be a Riemannian manifold. A diffeomorphism $F: M \to M$ is called a (Riemannian) *isometry* if $F^*g = g$. Equivalently, the condition is that

$$g_p(v,w) = g_{F(p)}(dF(v), dF(w))$$

for all $p \in M$ and $v, w \in T_pM$.

Show that each of the following maps defines an isometry of the hyperbolic plane $\mathbb{H}^2$. (Recall $\mathbb{H}^2$ denotes the manifold $\{(x,y) \mid y > 0\}$ with Riemannian metric $g_{\mathbb{H}^2} = \frac{1}{y^2}(dx^2 + dy^2)$.)

(a) $(x,y) \mapsto (x+c, y)$, where $c \in \mathbb{R}$ is a constant.
(b) $(x,y) \mapsto (e^t x, e^t y)$, where $t \in \mathbb{R}$ is a constant.
(c) $(x,y) \mapsto \left(\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2}\right)$

Then use these results to show:
(d) For all $p, q \in \mathbb{H}^2$, there exists an isometry $F: \mathbb{H}^2 \to \mathbb{H}^2$ such that $F(p) = q$.

(2) Lee, Chapter 12, Problem 7, on page 325.

(3) Lee, Chapter 14, Problem 5, on page 375. You may assume M has no boundary.

(4) Lee, Chapter 14, Problem 6, parts (a), (c), and (d), on page 375.

(5) Let $\mathbb{S}^2$ be the unit sphere in $\mathbb{R}^3$ and let η denote the element of $\Omega^2(\mathbb{S}^2)$ defined by $\eta_p(v,w) = \det(p,v,w)$. Here (p,v,w) refers to the 3×3 matrix with rows p, v, and w. Determine the explicit coordinate expression for $(\sigma^{-1})^*(\eta)$, where $\sigma: \mathbb{S}^2 \setminus \{N\} \to \mathbb{R}^2$ is the stereographic projection chart discussed in problem 7 of chapter 1 (on page 30).

Your answer should include an explicit function $f(x_1, x_2)$ so that

$$(\sigma^{-1})^*(\eta) = f(x_1,x_2)\, dx_1 \wedge dx_2.$$

[ND]

(6) Lee, Chapter 15, Problem 1, on page 397.

(7) Let G be a Lie group. Show that G is an orientable manifold. [ND]