Math 549: Differentiable Manifolds I – David Dumas – Fall 2017

Homework 7

Due Monday, October 30 at 1:00pm

(1) Lee, Chapter 11, Problem 7, on page 300.

(2) Let G be a Lie group with Lie algebra $\mathfrak{g}$. Recall that $\mathfrak{g}$ is the subalgebra of $\mathrm{Vect}(G)$ consisting of left-invariant vector fields. Show that the following vector spaces are naturally isomorphic to one another, and describe the isomorphisms explicitly:

- $\mathfrak{g}^* = \mathrm{Hom}(\mathfrak{g}, \mathbb{R})$, the linear dual of the vector space underlying $\mathfrak{g}$.
- The subspace of $\Omega^1(G)$ consisting of left-invariant 1-forms, i.e. the space of all 1-forms ω such that $L_g^*\omega = \omega$ for all $g \in G$.
- The cotangent space at the identity, T_e^*G.

(3) Let $G = \mathrm{GL}_2\mathbb{R}$, and let $\omega_{i,j} \in \Omega^1(G)$ denote the left-invariant 1-form on G which corresponds, by the isomorphism constructed in the previous problem, to the element of $T_e^*G \simeq \mathrm{Hom}(\mathrm{M}_{2\times 2}(\mathbb{R}), \mathbb{R})$ given by $A \mapsto A_{i,j}$ (the linear form giving the i,j matrix entry).

Compute $\displaystyle\int_\gamma \omega_{i,j}$ for each of the following oriented curves in G:

(a) $\gamma(t) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$ for $t \in [0,1]$.

(b) $\gamma(t) = \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix}$ for $t \in [0,1]$.

(c) $\gamma(t) = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}$ for $t \in [0,2\pi]$.

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(4) Lee, Chapter 13, Problem 4, on page 345.

(5) Recall $\mathbb{H}^2$ denotes the Riemannian manifold whose underlying smooth manifold is $\{(x,y) \in \mathbb{R}^2 \mid y > 0\}$ with Riemannian metric $g_{\mathbb{H}^2} = \frac{1}{y^2} g_{\mathbb{R}^2}$ where $g_{\mathbb{R}^2}$ is the Euclidean metric (which Lee denotes by $\bar{g}$ in Example 13.1).

Show that the Riemannian distance function $d_{\mathbb{H}^2}$ satisfies

$$d_{\mathbb{H}^2}((0,1),(0,y)) = \log y.$$

(Hint: A useful intermediate step would be to show that the paths $\gamma(t) = (x(t), y(t))$ and $\gamma^{\mathrm{vert}}(t) = (0, y(t))$ satisfy $L(\gamma^{\mathrm{vert}}) \leq L(\gamma)$.