Math 549: Differentiable Manifolds I – David Dumas – Fall 2017

Homework 5

Due Monday, October 9 at 1:00pm

(1) Consider the binary operation on $\mathbb{R}^2$ defined by the formula

$$(x,y)\cdot(x',y') = (x+x', y+e^x y').$$

(a) Show that this gives a Lie group structure on $\mathbb{R}$ with identity element $e = (0,0)$. For the rest of this problem, let G denote this Lie group.

(b) Show that G is not isomorphic (as a Lie group) to $(\mathbb{R}^2, +)$.

(c) Sketch the following vector fields on G:

(i) The extension of $\left.\frac{\partial}{\partial x}\right|_e$ to a left-invariant vector field.

(ii) The extension of $\left.\frac{\partial}{\partial x}\right|_e$ to a right-invariant vector field.

(iii) The extension of $\left.\frac{\partial}{\partial y}\right|_e$ to a left-invariant vector field.

(iv) The extension of $\left.\frac{\partial}{\partial y}\right|_e$ to a right-invariant vector field.

(Label or caption each sketch so it is clear which vector field is shown!)

(2) Lee, Chapter 8, Problem 10, on page 201. (Computing a pushfoward.)

(3) Let G be a Lie group and $a \in G$. Let $I_a : G \to G$ be defined by $I_a(g) = aga^{-1}$. Show that I_a is a Lie group automorphism.

(4) Lee, Chapter 7, Problem 13, on page 172. (The unitary group.)

(5) Lee, Chapter 7, Problem 16, on page 172. ($\mathrm{SU}(2) \simeq \mathbb{S}^3$)

(6) Consider $\mathbb{S}^3$ as the unit sphere in $\mathbb{C}^2$, i.e. $\mathbb{S}^3 = \{(z_1,z_2) \in \mathbb{C}^2 \,|\, |z_1|^2 + |z_2|^2 = 1\}$. Define an action of $\mathbb{S}^1$ on $\mathbb{S}^3$ by $\zeta \cdot (z_1,z_2) = (\zeta z_1, \zeta z_2)$.

(a) Show that this is a smooth action.

(b) Show that the orbits are circles (i.e. each orbit is an embedded submanifold diffeomorphic to $\mathbb{S}^1$).

(c) Show that two points of $\mathbb{S}^3$ lie in the same orbit of this action if and only if they span the same 1-dimensional complex subspace of $\mathbb{C}^2$.

This action of $\mathbb{S}^1$ on $\mathbb{S}^3$ is called the *Hopf action*.

(7) Let $\Theta : \mathbb{R} \times \mathbb{S}^3 \to \mathbb{S}^3$ be defined by $\Theta(t,(z_1,z_2)) = (e^{it}z_1, e^{it}z_2)$.

(a) Show that this is a smooth flow on $\mathbb{S}^3$.

(b) Calculate the vector field $V \in \mathrm{Vect}(\mathbb{S}^3)$ the is the infinitesimal generator of this flow. Write your answer in terms of the realization of $T_{(z_1,z_2)}\mathbb{S}^3$ as a subspace of $T_{(z_1,z_2)}\mathbb{C}^2$ (e.g. in terms of the basis $\frac{\partial}{\partial x_k}, \frac{\partial}{\partial y_k}$ where $z_k = x_k + iy_k$).

(c) Show that the vector field V does not vanish at any point of $\mathbb{S}^3$.

This flow on $\mathbb{S}^3$ is called the *Hopf flow*.