Math 549: Differentiable Manifolds I – David Dumas – Fall 2017

Homework 3

Due Monday, September 25 at 1:00pm

(1) Suppose S is a *compact* smooth manifold and $F : S \to M$ is an injective immersion. Show that F is a smooth embedding.

(2) Let $\Pi : \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{RP}^n$ be the map defined by $\Pi(v) = \mathrm{span}(v)$. Show that Π is a smooth submersion.

(3) — *This problem has been deleted from the assignment.* —

(4) Read Example 1.36 on the construction of a smooth structure on $G_k(V)$, the set of k-dimensional linear subspaces of an n-dimensional vector space V.

Let $\mathrm{GL}(n, \mathbb{R})$ denote the set of invertible $n \times n$ matrices with real entries. This is a smooth manifold because it is an open submanifold of $\mathbb{R}^{n^2}$.

Let C_k denote the map $\mathrm{GL}(n, \mathbb{R}) \to G_k(\mathbb{R}^n)$ defined by

$$C_k(M) = \mathrm{span}\,(\text{First } k \text{ columns of } M)$$

Show that C_k is a well-defined smooth submersion.

(5) Let $S \subset M$ be an embedded submanifold. Then for any $f \in C^\infty(M)$, the restriction $f|_S$ is a smooth function on S, because it is the composition of f with the embedding $\iota : S \to M$, both of which are smooth maps by hypothesis.

Show that in general, not all smooth functions on S arise this way.

That is, construct an example of a manifold M, an embedded submanifold S, and a smooth function $g \in C^\infty(S)$ that cannot be expressed as the restriction of a smooth function on M. [ND]

(6) Lee, Chapter 4, Problem 6, on page 96.

(7) Lee, Chapter 5, Problem 7, on page 123. Hint: You may use the main theorem about fibers of submersions in your solution, but remember that it only gives a *sufficient* condition for a level set to be an embedded submanifold.