Math 549: Differentiable Manifolds I – David Dumas – Fall 2017

Homework 2

Due Monday, September 18 at 1:00pm

(1) Let M and N be smooth manifolds, with M connected. Let $F : M \to N$ be a smooth map such that dF_p is the zero map for each $p \in M$. Show that F is a constant map. [LEE]

(2) Lee, Chapter 3, Problem 5. (You can't square the circle with a diffeomorphism.)

(3) Let M be a smooth n-manifold, $p \in M$, and let $C_p^\infty(M)$ denote the space of germs of smooth functions at p (as defined on page 71 of the text). Let $\mathfrak{m}_p \subset C_p^\infty(M)$ denote the set of germs $[(f,U)]$ such that $f(p) = 0$.

(a) Show that $\mathfrak{m}_p$ is a maximal ideal of the ring $C_p^\infty(M)$.

(b) Let $\mathfrak{m}_p^2 \subset C_p^\infty(M)$ denote the set of finite linear combinations of germs of the form $[(fg,U)]$ where $[(f,U)] \in \mathfrak{m}_p$ and $[(g,U)] \in \mathfrak{m}_p$. Note that $\mathfrak{m}_p^2 \subset \mathfrak{m}_p$, and let $\mathfrak{m}_p/\mathfrak{m}_p^2$ denote the quotient vector space.

Show that T_pM is isomorphic to the dual space $(\mathfrak{m}_p/\mathfrak{m}_p^2)^*$, where the isomorphism

$$T_pM \to (\mathfrak{m}_p/\mathfrak{m}_p^2)^*$$

is given by $v \mapsto \xi_v$ where $\xi_v : \mathfrak{m}_p/\mathfrak{m}_p^2 \to \mathbb{R}$ denotes the map

$$\xi_v([(f,U)] + \mathfrak{m}_p^2) = v(f).$$

(In particular you must show that ξ_v is well-defined.)

(4) Let $F : \mathbb{R} \to \mathbb{R}^2$ be defined by $F(t) = (e^t, \sin t)$. Show that F is an injective smooth immersion. Determine (with proof) whether or not F is a smooth embedding. Also sketch the image of F.

(5) Construct two injective smooth immersions $F_1, F_2 : \mathbb{R} \to \mathbb{R}^2$ that have the same image, but which are "essentially different" in the following sense: There does not exist a diffeomorphism $\varphi : \mathbb{R} \to \mathbb{R}$ such that $F_1 = F_2 \circ \varphi$. [BOOTHBY]

(6) Does there exist a smooth immersion from $\mathbb{R}$ to $\mathbb{R}^2$ whose image contains $\mathbb{Z}^2$? (Either construct one, or show that no such immersion exists.)

(7) Lee, Chapter 4, problem 10, on page 96.