Math 549: Differentiable Manifolds I – David Dumas – Fall 2017

Homework 10

Due Monday, November 20 at 1:00pm

(1) Show that the diagram (14.27) on page 368 of the textbook commutes, so that the operators grad (gradient), curl, and div (divergence) as studied in vector calculus on $\mathbb{R}^3$ are all equivalent to specific instances of the exterior derivative for differential forms.

(2) Use the commutative diagram of the previous problem to show that Stokes' theorem for differential forms implies all of the following results from 2- and 3-dimensional vector calculus:

(a) The fundamental theorem of calculus for line integrals of gradient vector fields.
(b) Green's theorem
(c) Stokes' theorem
(d) The divergence theorem

You may take statements of the vector calculus theorems from a multivariable calculus text such as Stewart or Strang (the latter being available free online).

Warning: For the purposes of this problem, the definition of the circulation of a vector field $\mathbf{F}$ along a curve $\Gamma = \{\gamma(t) \mid a \leq t \leq b\}$ is

$$\int_a^b \mathbf{F}(\gamma(t)) \cdot \gamma'(t)\, dt.$$

There is another notation for this integral that is often used in vector calculus courses, and which looks like the integral of a differential form. Do not use that notation unless you first show that it is unambiguous, i.e. that the circulation as defined above is equal to the integral of a certain 1-form.

(3) Show that the Lie derivative on forms satisfies

$$L_X(\alpha \wedge \beta) = L_X(\alpha) \wedge \beta + \alpha \wedge L_X(\beta).$$

(After completing this problem, you might want to think about how and why this formula differs from the analogous one for the interior product, i.e. Lemma 14.13(b) on page 358 of the textbook. That's just a suggestion; it is not an assignment or hint.)

(4) Suppose M is an oriented compact smooth manifold with boundary. Let $i : \partial M \to M$ denote the inclusion of the boundary. Show that there does *not* exist a smooth map $f : M \to \partial M$ such that $f \circ i = \mathrm{Id}_{\partial M}$. [Lee]

Hint: Use Stokes' theorem and an orientation form on ∂M.

(5) Let M be a compact orientable manifold of dimension n, and let $\eta \in \Omega^{n-1}(M)$. Show that there exists a point $p \in M$ such that $(d\eta)_p = 0$.

(6) Let M be a smooth manifold and let $X \in \mathrm{Vect}(M)$ be a smooth and *nowhere vanishing* vector field. Let $i_X^k : \Omega^k(M) \to \Omega^{k-1}(M)$ denote the interior product operator on k-forms, with the convention that $\Omega^{-1}(M) = \{0\}$ and that i_X^0 is the zero linear map. Show that for any $k \geq 0$ we have $\ker(i_X^k) = \mathrm{im}(i_X^{k+1})$.