Math 535: Complex Analysis – Spring 2016 – David Dumas

Practice Final Exam

- Complete **five** of the problems below.
- Each problem is worth 10 points.
- If you complete more than three problems (which is *not* recommended) your score will be the sum of your five best problem scores.

Problems:

(1) Compute

$$\oint_{S^1} \frac{dz}{25600z - z^3 + z^5 - 99z^9}$$

where S^1 denotes the unit circle $\{z : |z| = 1\}$ with the counter-clockwise orientation.

(2) Let

$$f_n(z) = \exp\left(-\left(\frac{z}{1} + \frac{z^2}{2} + \cdots + \frac{z^n}{n}\right)\right).$$

(a) Show that f_n converges locally uniformly on $\Delta = \{z : |z| < 1\}$, and identify the limit function.

(b) Does f_n converge locally uniformly on $|z| < 2$?

(3) Find the Laurent expansion for the function $\dfrac{12}{z^2(z+1)(z-2)}$ in the annulus $1 < |z| < 2$.

(4) Compute $\displaystyle\int_0^\infty \frac{x^2\,dx}{x^4 + 5x^2 + 4}$.

(5) Does there exist an entire function f with no zeros and so that the *real* solutions of the equation $f(x) = 1$ are exactly the prime numbers? (That is, $f(p) = 1$ for each prime $p \in \mathbb{N}$, and if $x \in \mathbb{R}$ is not a prime, then $f(x) \neq 1$.)

Either construct such a function or prove that no such function exists.

(6) Construct a conformal mapping $f : \lhd \to \Omega$ where

$$\lhd = \{z : 0 < |z| < 1, |\arg(z)| < \frac{\pi}{8}\}$$

and

$$\Omega = \mathbb{H} \setminus \{iy : y \in (0, 535]\}.$$

(7) Can a (real-valued) harmonic function on an open set in $\mathbb{C}$ have an isolated zero? Offer an example or a proof that it is impossible.

(8) Write a formula for a conformal mapping from the upper half plane to an equilateral triangle of unit side length.