**Math 535: Complex Analysis – Spring 2016 – David Dumas**

# Practice Final Exam Solutions

- Complete **five** of the problems below.
- Each problem is worth 10 points.
- If you complete more than three problems (which is *not* recommended) your score will be the sum of your five best problem scores.

**Problems:**

(1) Compute

$$\oint_{S^1} \frac{dz}{25600z - z^3 + z^5 - 99z^9}$$

where $S^1$ denotes the unit circle $\{z : |z| = 1\}$ with the counter-clockwise orientation.

**Solution.** We use the residue theorem. The integrand has a simple pole at the origin, and we must determine whether it has any other poles in the unit disk. Let $f(z) = 25600z - z^3 + z^5 - 99z^9$ denote the denominator, and let

$$g(z) = 25600z - 100z^9 = 100z(256 - z^8)$$

which is a polynomial with a root at $z = 0$ and all of its other roots on $|z| = 2$. Since

$$f(z) = g(z) + z^9 + z^5 - z^3$$

we find for $|z| = 1$ that

$$|f(z) - g(z)| \leqslant 3$$

whereas

$$|f(z)| \geqslant \big||25600z| - |99z^9|\big| \geqslant 25501$$

also for $|z| = 1$. Thus $|f - g| < |f|$ on $S^1$, and by Rouché's theorem, $f$ and $g$ have the same number of roots in the unit disk, i.e. one.

Therefore the integral we want to compute is equal to

$$2\pi i \operatorname{Res}_{z=0} \frac{1}{f(z)} = 2\pi i \lim_{z\to 0} \frac{z}{f(z)} = 2\pi i \lim_{z\to 0} \frac{1}{25600 - z^2 + z^4 - 99z^8} = \frac{\pi i}{12800}.$$

(2) Let

$$f_n(z) = \exp\left(-\left(\frac{z}{1} + \frac{z^2}{2} + \cdots + \frac{z^n}{n}\right)\right).$$

(a) Show that $f_n$ converges locally uniformly on $\Delta = \{z : |z| < 1\}$, and identify the limit function.

(b) Does $f_n$ converge locally uniformly on $|z| < 2$?

**Solution.**
(a) The Taylor series for the principal branch of $\log(1-z)$ is $-\sum_{k=1}^{\infty} \frac{z^k}{k}$, and $f_n = \exp(s_n)$ where $s_n$ is the $n^{\text{th}}$ partial sum of this series. Since $\log(1-z)$ is holomorphic in $\Delta$, we have

$s_n \to \log(1-z)$ locally uniformly in this disk. Thus if we can show that locally uniform convergence is preserved by composition with exp, it will follow that

$$f_n \to \exp(\log(1-z)) = 1-z$$

on $\Delta$.

On any closed disk $D$ in $\mathbb{C}$ there is a constant $M$ such that $|\exp(z)-\exp(w)| \leqslant M|z-w|$; in fact, we can take $M = \sup_{z\in D} |\exp(z)|$. For any such $D$ contained in $\Delta$ and any $\varepsilon > 0$ we therefore have

$$|f_n(z)-(1-z)| = |\exp(s_n(z)) - \exp(\log(1-z))| < \varepsilon$$

for all $z \in D$ once we take $n$ large enough that $|s_n(z)-\log(1-z)| < \varepsilon/M$, which is possible by the uniform convergence of $s_n$ on $D$. Thus $f_n \to (1-z)$ locally uniformly on $\Delta$.

(b) No. If $f_n$ had a locally uniform limit on $\{z : |z| < 2\}$ then the limit would be a holomorphic function equal to $1-z$ on $\Delta$, and hence everywhere. However $1-z$ has an isolated zero at $z=1$, and by Hurwitz's theorem it cannot be the locally uniform limit of the sequence of nowhere-vanishing functions $f_n = \exp(s_n)$.

(3) Find the Laurent expansion for the function $\dfrac{12}{z^2(z+1)(z-2)}$ in the annulus $1 < |z| < 2$.

**Solution.** The annulus is centered at zero, so this Laurent expansion will consist of powers of $z = (z-0)$. One can proceed by the general formula for coefficients or by partial fraction decomposition. We choose the latter.

Notice that

$$\frac{12}{z^2(z+1)(z-2)} = \frac{3(z-2)}{z^2} - \frac{4}{z+1} + \frac{1}{z-2}.$$

In $|z| > 1$ we have

$$\begin{aligned} -\frac{4}{z+1} &= -\frac{4}{z}\frac{1}{1+z^{-1}} = -\frac{4}{z}\left(1 - z^{-1} + z^{-2} - z^{-3} + \cdots\right) \\ &= -4z^{-1} + 4z^{-2} - 4z^{-3} + 4z^{-4} - \cdots, \end{aligned}$$

where the expression in parentheses is a convergent geometric series. Similarly, using $|z| < 2$ we have

$$\begin{aligned} \frac{1}{z-2} &= -\frac{1}{2}\left(\frac{1}{1-\frac{z}{2}}\right) = -\frac{1}{2}\left(1 + \frac{1}{2}z + \frac{1}{4}z^2 + \frac{1}{8}z^3 + \cdots\right) \\ &= -\frac{1}{2} - \frac{1}{4}z - \frac{1}{8}z^2 - \frac{1}{16}z^3 - \cdots \end{aligned}$$

Finally we expand

$$\frac{3(z-2)}{z^2} = -\frac{6}{z^2} + \frac{3}{z}$$

Adding these series we find:

$$\frac{12}{z^2(z+1)(z-2)} = \left(\sum_{k=-\infty}^{-3} (-1)^k 4z^k\right) - 2z^{-2} - z^{-1} + \left(\sum_{k=0}^{\infty} \frac{-1}{2^{k+1}} z^k\right)$$

(4) Compute $\int_0^\infty \frac{x^2\,dx}{x^4+5x^2+4}$.

**Solution.** We convert to a contour integral and use residues. Let

$$f(z) = \frac{z^2}{z^4+5z^2+4} = \frac{z^2}{(z^2+1)(z^2+4)}$$

be the integrand and and let

$$I = \int_0^\infty f(x)dx$$

denote the integral in question. Since $f(x)$ is even we have $2I = \int_{-\infty}^{\infty} f(x)dx$.

Let $D_R$ denote the closed contour in $\mathbb{C}$ that is the concatenation of the real interval $[-R,R]$, oriented in the increasing direction, and the counterclockwise orientation of the upper semicircle on $|z| = R$. Denote the latter semicircle by $C_R$. Then we have

$$\oint_{D_R} f(z)dz = \int_{-R}^{R} f(x)dx + \int_{C_R} f(z)dz.$$

Since $f$ has a zero of order 2 at infinity, for large $|z|$ it is bounded by $M/|z|^2$, where $M$ is a constant. Thus for large $R$ we have

$$\left|\int_{C_R} f(z)dz\right| \leqslant \pi R \frac{M}{R^2}$$

which goes to zero as $R \to \infty$, hence

$$\lim_{R\to\infty} \oint_{D_R} f(z)dz = \lim_{R\to\infty} \int_{-R}^{R} f(x)dx = 2I$$

The left hand side is constant for $R$ large enough and is equal to the sum of residues in the upper half plane. Specifically, we find

$$I = \pi i(\mathrm{Res}_{z=i} f(z) + \mathrm{Res}_{z=2i} f(z))$$

since $z = i$ and $z = 2i$ are the poles in the upper half plane. Both of these poles are simple, so we have

$$\begin{aligned}\mathrm{Res}_{z=i} f(z) &= \lim_{z\to i}(z-i)f(z) = \lim_{z\to i} \frac{z^2}{(z+i)(z^2+4)} \\ &= \frac{i^2}{(2i)(i^2+4)} = \frac{i}{6}\end{aligned}$$

and

$$\begin{aligned}\mathrm{Res}_{z=2i} f(z) &= \lim_{z\to 2i}(z-2i)f(z) = \lim_{z\to 2i} \frac{z^2}{(z^2+1)(z+2i)} \\ &= \frac{4i^2}{(4i^2+1)(2i+2i)} = \frac{-i}{3}.\end{aligned}$$

Substituting, we find

$$I = \pi i(\frac{i}{6} - \frac{i}{3}) = \frac{\pi}{6}.$$

(5) Does there exist an entire function $f$ with no zeros and so that the *real* solutions of the equation $f(x) = 1$ are exactly the prime numbers? (That is, $f(p) = 1$ for each prime $p \in \mathbb{N}$, and if $x \in \mathbb{R}$ is not a prime, then $f(x) \neq 1$.)

Either construct such a function or prove that no such function exists.

**Solution.** We will construct such a function $f$. Suppose $g$ is an entire function which has zeros only at the primes, and which is real on $\mathbb{R}$. Then $f = \exp(g)$ has the desired properties:

- As the exponential of a function, $f$ is never zero
- The equation $f(x) = 1$ is equivalent to $g(x) = 2\pi ik$ and $k \in \mathbb{Z}$. However, since $g(x) \in \mathbb{R}$ for $x \in \mathbb{R}$, the only possibility for such $x$ is $k = 0$, and the only zeros of $g$ are the primes.

The construction of $g$ is easily accomplished by the Weierstrass factorization theorem. Since

$$\sum_{p \text{ prime}} \frac{1}{p^2} < \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty,$$

the infinite product of genus one

$$g(z) = \prod_{p \text{ prime}} \left(1 - \frac{z}{p}\right) \exp\left(\frac{z}{p}\right)$$

defines an entire function whose zero set is the set of primes. Furthermore, each factor in the product is real when $z \in \mathbb{R}$, hence $g$ is real on $\mathbb{R}$.

(6) Construct a conformal mapping $f : \lhd \to \Omega$ where

$$\lhd = \{z : 0 < |z| < 1, |\arg(z)| < \frac{\pi}{8}\}$$

and

$$\Omega = \mathbb{H} \setminus \{iy : y \in (0, 535]\}.$$

**Solution.** First consider the function $h(z) = z^8$, which satisfies $\arg(h(z)) = 8\arg(z)$ and therefore maps $\lhd$ conformally the slit disk

$$\Delta' = \{z : 0 < |z| < 1, \arg(z) \in (-\pi, \pi)\} = \Delta \setminus \{x \in \mathbb{R}, x \leqslant 0\}$$

Now we apply a Möbius transformation to map the disk to the upper half-plane, chosen so as to map the slit of $\Delta'$ to the correct interval on the imaginary axis. Specifically, let

$$g(z) = i\frac{1+z}{1-z}.$$

Since $g(-1) = 0$, $g(i) = -1$, $g(1) = \infty$ we have that $g$ maps the unit circle to the real axis. Also, $g(0) = i$ shows that the unit disk maps to the upper half-plane, and that the interval $(-1, 0]$ on $\mathbb{R}$ maps to a line or circular arc in $\mathbb{H}$ with endpoints 0 and $i$. Calculating

$$g(\bar{z}) = i\frac{1+\bar{z}}{1-\bar{z}} = \overline{\left(-i\frac{1+z}{1-z}\right)} = -\overline{g(z)}$$

we find that $z = \bar{z}$ implies $g(z) = -\overline{g(z)}$, that is, the correspondence $w = g(z)$ maps real $z$ to purely imaginary $w$. Thus the slit $(-1, 0]$ of $\Delta'$ corresponds by $g$ to the line segment

$\{iy : y \in (0,1]\}$. Finally, multiplying by 535 preserves $\mathbb{H}$ and transforms this segment to the one in the definition of $\Omega$.

Composing these operations, we find

$$f(z) = 535g(h(z)) = 535i\frac{1+z^8}{1-z^8}$$

is a map with the desired properties.

(7) Can a (real-valued) harmonic function on an open set in $\mathbb{C}$ have an isolated zero? Offer an example or a proof that it is impossible.

**Solution.** No, this is impossible. Suppose $u(a) = 0$ were isolated. Then for sufficiently small $\rho$ we have that $\{z : |z-a| \leqslant \rho\}$ is contained in the domain of $u$ and that $g(\theta) = u(a+\rho e^{i\theta})$ is nonzero for all $\theta$. Since $g(\theta)$ is a continuous function of $\theta$ with no zeros, it is either everywhere positive or everywhere negative. In either case we conclude $\int_0^{2\pi} g(\theta)d\theta \neq 0$. However, by the mean value property of harmonic functions we have

$$0 = u(a) = \frac{1}{2\pi}\int_0^{2\pi} g(\theta)d\theta,$$

a contradiction.

(8) Write a formula for a conformal mapping from the upper half plane to an equilateral triangle of unit side length.

**Solution.** By the Schwarz-Christoffel theorem, for any base point $z_0 \in \mathbb{H}$ the mapping

$$g(z) = \int_{z_0}^{z} \frac{d\zeta}{\zeta^{\frac{2}{3}}(\zeta-1)^{\frac{2}{3}}}$$

is conformal onto a triangle in $\mathbb{C}$ with internal angles $(\pi/3, \pi/3, \pi/3)$ at vertices corresponding to $(0, 1, \infty) \in \partial\mathbb{H}$. Equiangular triangles are equilateral, so we need only multiply by a suitable real constant so that the side length is 1.

The formula above gives that the side length of the image triangle is

$$|g(1)-g(0)| = \left|\int_0^1 \frac{dx}{x^{\frac{2}{3}}(x-1)^{\frac{2}{3}}}\right| = \int_0^1 \frac{dx}{x^{\frac{2}{3}}(1-x)^{\frac{2}{3}}},$$

where the last equality holds by factoring out the constant $(-1)^{-\frac{2}{3}}$ of modulus one, leaving a positive integrand. Thus we find

$$f(z) = \frac{\int_{z_0}^{z} \frac{d\zeta}{\zeta^{\frac{2}{3}}(\zeta-1)^{\frac{2}{3}}}}{\int_0^1 \frac{dx}{x^{\frac{2}{3}}(1-x)^{\frac{2}{3}}}}$$

has the desired properties.

**Remark.** It can be shown that $\displaystyle\int_0^1 \frac{dx}{x^{\frac{2}{3}}(1-x)^{\frac{2}{3}}} = \frac{\Gamma(\frac{1}{3})\Gamma(\frac{1}{6})}{2^{\frac{2}{3}}\sqrt{\pi}}$.