Math 535: Complex Analysis – Spring 2016 – David Dumas

Final Exam

Instructions:

- For full credit, you must solve **five** of the problems below.
- No books or notes are allowed at the exam.
- In your solutions you can use any theorems from the textbook or from the lectures, but in doing so you *must* make it clear what result is being used.

Problems:

(1) Calculate: $\displaystyle\int_0^\infty \frac{x^{\frac{1}{5}}}{x^2+1}\,dx$

(2) Does there exist an entire function f with the following properties?

- $f(1) = 0$
- $f(2) = 0$
- $f(z) \in \mathbb{R}$ if and only if $z \in \mathbb{R}$

Either give an example of such a function, or prove that no such function exists.

(3) For $n \in \mathbb{N}$ let $\Lambda_n \subset \mathbb{C}$ denote the lattice generated by $\omega_1 = 1$ and $\omega_2 = ni$. Let $\wp_n$ denote the Weierstrass function of Λ_n. Identify the limit of the meromorphic functions $\wp_n$ as $n \to \infty$, and the region on which the convergence is locally uniform.

(4) Suppose f is a holomorphic function on $|z| < 2$ that is *even* (that is, $f(-z) = f(z)$). Show that there exists a holomorphic function F on the annulus $1 < |z| < 2$ such that

$$F'(z) = \frac{f(z)}{z^2 - 1}.$$

(5) Completely describe the convergence of the power series

$$\sum_{n=1}^{\infty} \frac{z^{2n}}{2^n n^3}$$

for $z \in \mathbb{C}$. That is, determine the set of all z for which the series converges, and separately, identify the largest open set in which the convergence is locally uniform.

(6) Find all linear fractional transformations T such that $T(1) = 1$, $T(3) = 3$, and $T(T(z)) = z$ for all z.

(7) Find all holomorphic functions on $\mathbb{C}^*$ that satisfy:

$$|f(z)| < |z| + \big|\log|z|\big|$$

(8) Determine whether or not each family of holomorphic functions on the unit disk is normal:

(a) $\mathcal{F}_1 = \{f : \Delta \to \mathbb{C} \ : \ f(z) \neq 0 \text{ for all } z \in \Delta\}$
(b) $\mathcal{F}_2 = \{f : \Delta \to \mathbb{C} \ : \ f(z) \notin [0,1] \text{ for all } z \in \Delta\}$
(c) $\mathcal{F}_3 = \{f : \Delta \to \mathbb{C} \ : \ |f(z)| > 1 \text{ for all } z \in \Delta\}$