**Math 320 – Linear Algebra – David Dumas – Fall 2016**

# Solution to 1.5.13a

Let $V$ be a vector space over a field of characteristic not equal to two. Let $u$ and $v$ be distinct vectors in $V$. Prove that $\{u,v\}$ is a linearly independent if and only if $\{u+v,u-v\}$ is linearly independent.

*Proof.* First suppose that $\{u,v\}$ is linearly independent. Suppose that we have a linear combination of $u+v$ and $u-v$ that is equal to the zero vector, i.e.

$$a(u+v)+b(u-v)=0$$

Then expanding and regrouping terms this becomes

$$(a+b)u+(a-b)v=0$$

By linear independence of $\{u,v\}$ we have that $a+b=0$ and $a-b=0$. Adding these equations we find $2a=0$. Since the characteristic of the field is not 2, we can multiply by $\frac{1}{2}$ to obtain $a=0$. Then from $a+b=0$ it follows also that $b=0$. We have therefore shown that the only linear combination $a(u+v)+b(u-v)$ that is equal to zero has $a=b=0$. Thus $\{u+v,u-v\}$ is linearly independent.

Now suppose that $\{u+v,u-v\}$ is linearly independent. Suppose that we have a linear combination of $u$ and $v$ that is equal to the zero vector, i.e.

$$au+bv=0$$

Since the characteristic of the field is not 2, the field contains an element $\frac{1}{2}$ and we can write

$$u=\frac{1}{2}(u+v)+\frac{1}{2}(u-v)$$

and similarly

$$v=\frac{1}{2}(u+v)-\frac{1}{2}(u-v)$$

Substituting these into the linear combination above we find

$$a\left(\frac{1}{2}(u+v)+\frac{1}{2}(u-v)\right)+b\left(\frac{1}{2}(u+v)-\frac{1}{2}(u-v)\right)=0$$

and after regrouping terms,

$$\frac{1}{2}(a+b)(u+v)+\frac{1}{2}(a-b)(u-v)=0.$$

Since $\{u+v,u-v\}$ is linearly independent we conclude $\frac{1}{2}(a+b)=0$ and $\frac{1}{2}(a-b)=0$. Adding these equations we find $a=0$, and subtracting them gives $b=0$. We have therefore shown that the only linear combination of $au+bv$ that is equal to zero has $a=b=0$. Thus $\{u,v\}$ is linearly independent. $\square$