Math 320 – Linear Algebra – David Dumas – Fall 2016

Exam 2 Solutions

(1) Give an example of a 4×3 matrix B over $\mathbb{R}$ so that
- B has rank 2, and
- All of the matrix entries of B are nonzero, and
- $Bx = 0$, where $x = (1,1,1)$, and
- $(1,2,3,4)$ is *not* in the range $R(L_B)$.

Of course you must prove that your example B has these properties.

Solution. There are of course many possible solutions, and many ways to find one. We give one example.

If the two of the rows of A are not proportional to one another, then the rank is at least two. If we make sure that among all of the rows of A, there are only two distinct ones, then the rank is at most two. Hence choosing two non-proportional rows and then duplicating them to fill the rest of the matrix will give a matrix satisfying $\operatorname{rank}(B) = 2$.

The condition that $Bx = 0$ says that the sum of the columns is equal to the zero vector (in $\mathbb{R}^4$). To ensure this we choose row vectors that sum to zero. We must also make sure the rows have no zero entries. One possibility is to use $(1,1,-2)$ as the first and second row, and $(1,-2,1)$ for the other rows. The resulting matrix is:

$$\begin{pmatrix} 1 & 1 & -2 \\ 1 & 1 & -2 \\ 1 & -2 & 1 \\ 1 & -2 & 1 \end{pmatrix}$$

We have constructed this to satisfy the first three conditions. It remains to see whether the column vector $(1,2,3,4)$ is in the range. We notice that each column of A has its first and second entry equal to one another, and thus the same is true for any linear combination of the columns. Since $(1,2,3,4)$ does *not* satisfy this equation, it cannot be a linear combination of the columns of A, i.e. $(1,2,3,4) \notin R(L_A)$.

(2) Determine how many 2×2 matrices over $\mathbb{Z}/2$ have rank zero, how many have rank one, and how many have rank two. For each of the invertible ones, determine the inverse.

Solution. The only matrix of rank zero is the matrix with all entries equal to zero, $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. Hence there is *one* matrix of rank zero.

A matrix of rank one has rows that are proportional to each other. In $\mathbb{Z}/2$, however, the only possible scalars of proportionality are 0 and 1. Hence a matrix of rank 1 either has one row equal to zero or it has its rows equal to one another. Thus we can enumerate some rank 1 matrices by considering all possible first rows: $(0,0)$, $(1,0)$, $(0,1)$, and $(1,1)$, and for each of them adding all compatible second row vectors.

For a matrix of rank 1 with first row $(0,0)$, the second row can be any nonzero vector, giving possibilities

$$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}.$$

For each of the other possibilities for the first row, the second row can either be zero or can be equal to the first row, giving possibilities

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$$

and

$$\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

Thus there are *nine* matrices of rank one.

The remaining matrices have rows that are both nonzero, and are not equal to one another. These have rank two, and hence are invertible. We enumerate these and also calculate the inverse of each. The inverse can be calculated by row operations, but it is also convenient to use the 2×2 explicit inverse formula, which in general is:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

However, for $\mathbb{Z}/2$ we have $-1 = 1$ and the only possibility for the nonzero scalar $ad-bc$ is 1, hence the formula becomes

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \begin{pmatrix} d & b \\ c & a \end{pmatrix}$$

for any matrix known to have rank 2. Considering the possible pairs of rows among $(1,0)$, $(0,1)$, and $(1,1)$, we find *six* invertible matrices:

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

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(3) Let $\mathbb{R}[x]_d$ denote the vector space of polynomials of degree at most d with real coefficients. Let $T : \mathbb{R}[x]_2 \to \mathbb{R}[x]_3$ denote the linear transformation defined by

$$T(p) = (x-2)\,p.$$

Thus for example $T(-2x+3) = -2x^2+7x-6$.

(a) Find the matrix of T relative to the standard ordered bases $\{1,x,x^2\}$ of $\mathbb{R}[x]_2$ and $\{1,x,x^2,x^3\}$ of $\mathbb{R}[x]_3$.

(b) Determine the rank and nullity of T.

Solution.

(a) We apply T to each of the basis vectors of $\mathbb{R}[x]_2$ and express the result in terms of the basis of $\mathbb{R}[x]_3$ to determine a column of the matrix.

For example $T(1) = x-2$ has coefficients $(-2,1,0,0)$, hence this is the first column of T. Calculating similarly for $T(x)$ and $T(x^2)$ we find the matrix of T is

$$A = \begin{pmatrix} -2 & 0 & 0 \\ 1 & -2 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix}$$

(b) The vectors $T(1)$, $T(x)$, and $T(x^2)$ are linearly independent because they are polynomials of different degrees; each has leading coefficient equal to 1, but for respective degrees 1, 2 and 3. Since these vectors span the range of L_A, we have

$$\operatorname{rank}(A) = \dim R(L_A) = 3.$$

By the dimension theorem we have

$$\operatorname{nullity}(A) = \dim \mathbb{R}[x]_2 - \operatorname{rank}(A) = 3-3 = 0.$$

(4) Let $\mathbb{R}[x]_2$ denote the vector space of polynomials of degree at most 2 with real coefficients. Suppose $a, b \in \mathbb{R}$ are given. Then

$$\beta = \{1, (x-a), (x-a)^2\} \text{ and } \beta' = \{1, (x-b), (x-b)^2\}$$

are ordered bases of $\mathbb{R}[x]_2$. Find the matrix Q of change of basis from β' to β.

Solution. To calculate a column of Q we need to express an element of basis β' as a linear combination of elements of basis β. With a bit of algebra we find:

$$\begin{aligned} 1 &= 1\cdot 1 + 0\cdot(x-a) + 0\cdot(x-a)^2 \\ x-b &= (a-b)\cdot 1 + 1\cdot(x-a) + 0\cdot(x-a)^2 \\ (x-b)^2 &= (a-b)^2\cdot 1 + 2(a-b)\cdot(x-a) + 1\cdot(x-a)^2 \end{aligned}$$

hence the matrix is

$$Q = \begin{pmatrix} 1 & a-b & (a-b)^2 \\ 0 & 1 & 2(a-b) \\ 0 & 0 & 1 \end{pmatrix}.$$

(5) Let M be the vector space of 2×2 matrices over $\mathbb{R}$. Consider the ordered basis

$$\beta = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\}$$

of M. Find the dual basis $\beta^* = \{f_1, f_2, f_3, f_4\}$ of M^* and then express the trace function $\mathrm{tr} : M \to \mathbb{R}$ as a linear combination of these dual basis vectors.

Solution. Let us denote the given basis vectors by

$$x_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \; x_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \; x_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \; x_4 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The defining property of the dual basis is $f_i(x_j) = \delta_{ij}$.

Let us solve for f_1. As a linear functional on M it can be expressed as

$$f_1(v) = f_1\left(\begin{pmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{pmatrix} \right) = av_{11} + bv_{12} + cv_{21} + dv_{22}$$

for some real scalars a, b, c, d which we must determine. The conditions $f_1(x_1) = 1$ and $f_1(x_2) = f_1(x_3) = f_1(x_4) = 0$ become the system of equations

$$\begin{aligned} a + d &= 1 \\ b &= 0 \\ c &= 0 \\ a - d &= 0 \end{aligned}$$

which has solution $a = d = \frac{1}{2}$ and $b = c = 0$. Hence for a 2×2 matrix v we have

$$f_1(v) = \tfrac{1}{2}v_{11} + \tfrac{1}{2}v_{22}.$$

Similarly each of f_2, f_3, f_4 can be found by solving a system of four linear equations in four variables, yielding

$$\begin{aligned} f_2(v) &= v_{12} \\ f_3(v) &= v_{21} \\ f_4(v) &= \tfrac{1}{2}v_{11} - \tfrac{1}{2}v_{22} \end{aligned}$$

Finally, we notice that $f_1(v) = \frac{1}{2}\mathrm{tr}(v)$, so the expression of the trace function as a linear combination of the dual basis is simply

$$\mathrm{tr} = 2f_1.$$