Math 320 – Linear Algebra – David Dumas – Fall 2016

Exam 2

Instructions:

- Solve **three** problems from the list below.
- Every answer must be justified. Usually that means you need to write a proof.
- You can use any proposition, lemma, or theorem that we covered in lecture thus far or that was in the corresponding sections of the textbook. However, you **must** clearly state what you are using.
- When writing your solutions, clearly label each one with the problem number.

(1) Give an example of a 4×3 matrix B over $\mathbb{R}$ so that
- B has rank 2, and
- All of the matrix entries of B are nonzero, and
- $Bx = 0$, where $x = (1,1,1)$, and
- $(1,2,3,4)$ is *not* in the range $R(L_B)$.

Of course you must prove that your example B has these properties.

(2) Determine how many 2×2 matrices over $\mathbb{Z}/2$ have rank zero, how many have rank one, and how many have rank two. For each of the invertible ones, determine the inverse.

(3) Let $\mathbb{R}[x]_d$ denote the vector space of polynomials of degree at most d with real coefficients. Let $T : \mathbb{R}[x]_2 \to \mathbb{R}[x]_3$ denote the linear transformation defined by

$$T(p) = (x-2)\,p.$$

Thus for example $T(-2x+3) = -2x^2+7x-6$.
(a) Find the matrix of T relative to the standard ordered bases $\{1,x,x^2\}$ of $\mathbb{R}[x]_2$ and $\{1,x,x^2,x^3\}$ of $\mathbb{R}[x]_3$.
(b) Determine the rank and nullity of T.

(4) Let $\mathbb{R}[x]_2$ denote the vector space of polynomials of degree at most 2 with real coefficients. Suppose $a,b \in \mathbb{R}$ are given. Then

$$\beta = \{1,(x-a),(x-a)^2\} \text{ and } \beta' = \{1,(x-b),(x-b)^2\}$$

are ordered bases of $\mathbb{R}[x]_2$. Find the matrix Q of change of basis from β' to β.

(5) Let M be the vector space of 2×2 matrices over $\mathbb{R}$. Consider the ordered basis

$$\beta = \left\{ \begin{pmatrix}1&0\\0&1\end{pmatrix}, \begin{pmatrix}0&1\\0&0\end{pmatrix}, \begin{pmatrix}0&0\\1&0\end{pmatrix}, \begin{pmatrix}1&0\\0&-1\end{pmatrix} \right\}$$

of M. Find the dual basis $\beta^* = \{f_1,f_2,f_3,f_4\}$ of M^* and then express the trace function $\mathrm{tr} : M \to \mathbb{R}$ as a linear combination of these dual basis vectors.