Math 320 – Linear Algebra – David Dumas – Fall 2016

Exam 1 Solutions

(1) Let $S = \{ (1,2,-1), (3,1,0), (0,-5,3) \} \subset \mathbb{R}^3$. Consider $\mathbb{R}^3$ as a vector space over $\mathbb{R}$.
  (a) Is S linearly independent?
  (b) Find a subset of S that is a basis for $\mathrm{span}(S)$.

Solution. (a) No, this set is linearly dependent. We have

$$3 \cdot (1,2,-1) - 1 \cdot (3,1,0) + 1 \cdot (0,-5,3) = (0,0,0).$$

More generally, solving the system of linear equations

$$a \cdot (1,2,-1) + b \cdot (3,1,0) + c \cdot (0,-5,3) = (0,0,0)$$

gives $b = -c$ and $a = 3c$, with c any scalar.

(b) Let $T = \{(1,2,-1),(3,1,0)\}$. Then by (a) we have that $(0,-5,3) \in \mathrm{span}(T)$, hence T generates $\mathrm{span}(S)$. We claim that T is linearly independent. To see this recall that a 2-element set $\{u,v\}$ is linearly independent if and only if neither of u,v is a scalar multiple of the other. The only scalar multiple of $(1,2,-1)$ that has third component zero is the zero vector, whereas all scalar multiples of $(3,1,0)$ have third component zero. Thus neither is a scalar multiple of the other. We have therefore shown T is linearly independent and that it generates $\mathrm{span}(S)$, so T is a basis for $\mathrm{span}(S)$.

(2) Let $\mathbb{R}[x]_2$ denote the vector space of polynomials of degree at most 2 with real coefficients, which is a vector space over $\mathbb{R}$. Let $T : \mathbb{R}[x]_2 \to \mathbb{R}^3$ be the linear transformation defined by

$$T(p) = (p(1), p(2), p(3)).$$

  (a) Find a basis of the null space of T.
  (b) Find a basis of the range of T.

Solution. There are several ways to do this problem. One convenient approach is to recall the Lagrange interpolation theorem, which gives a basis $\{p_1, p_2, p_3\}$ of $\mathbb{R}[x]_2$ such that $T(p_1) = (1,0,0)$, $T(p_2) = (0,1,0)$ and $T(p_3) = (0,0,1)$. Since $\{(1,0,0),(0,1,0),(0,0,1)\}$ generates $\mathbb{R}^3$, this shows that the range of T is $\mathbb{R}^3$, and $\mathrm{rank}(T) = 3$. By the dimension theorem, $\mathrm{nullity}(T) = 3-3 = 0$, and we conclude $N(T) = \{0\}$. Thus:

(a) The empty set is a basis for the zero vector space $N(T) = \{0\}$.

(b) The standard basis $\{(1,0,0),(0,1,0),(0,0,1)\}$ is a basis for $R(T) = \mathbb{R}^3$.

(3) Let $V = (\mathbb{Z}/2)^4$, a vector space over the field $\mathbb{Z}/2$.
  (a) How many elements does V have?
  (b) Let $W \subset V$ be the set of all vectors that have an even number of entries equal to 1. Show that W is a subspace of V and find a generating set for W.

Solution. (a) Since $\mathbb{Z}/2$ has 2 elements, the Cartesian product $(\mathbb{Z}/2)^4$ has $2^4 = 16$ elements.

(b) The problem can be approached directly by considering sums and scalar multiples and dividing into several cases (depending on how many entries of the vectors are nonzero).

An easier way is to notice that $(a,b,c,d) \in W$ if and only if $a+b+c+d = 0$. To prove this, first suppose $(a,b,c,d) \in W$. Then the sum $a+b+c+d$ has an even number of terms equal to 1, and grouping them in pairs and using $1+1=0$ we find that $a+b+c+d=0$. Now suppose $(a,b,c,d) \notin W$. Then the sum $a+b+c+d$ has an odd number of terms equal to 1, say $2k+1$ of them. We can separate the sum into its zero terms, k pairs $(1+1)$, and a single 1. Using $1+1=0$ on the pairs, we find the sum is equal to 1. Contrapositively we conclude that if $a+b+c+d=0$ then $(a,b,c,d) \in W$.

Using this characterization, we find that the set W a subspace because a scalar multiple of a vector (a,b,c,d) satisfying $a+b+c+d=0$ also satisfies that equation, and similarly for a sum of two vectors satisfying the equation.

(An equivalent way to explain the proof above would be: Define $T : (\mathbb{Z}/2)^4 \to \mathbb{Z}/2$ by $T(a,b,c,d) = a+b+c+d$. This is a linear transformation, and W is its null space.)

Finally, for a generating set we could take the entire subspace W. (The problem did not ask for a finite generating set or for a basis, though in this case, it happens that W is a finite set anyway.) With a bit more work, one could also show that W is generated by the three vectors

$$\{(1,1,0,0),(0,1,1,0),(0,0,1,1)\}$$

which form a basis.

(4) Suppose V is a vector space and W_1 and W_2 are subspaces of V. Let U be the set of all vectors in V that can be written as a sum of a vector in W_1 and a vector in W_2. That is,

$$U = \{w_1 + w_2 \mid w_1 \in W_1, w_2 \in W_2\}$$

Show that U is a subspace of V.

Solution. (a) Suppose that $u \in U$ and $c \in \mathbb{F}$. Then we can write $u = w_1 + w_2$ for some $w_1 \in W_1$ and $w_2 \in W_2$. Therefore

$$cu = c(w_1 + w_2) = cw_1 + cw_2.$$

Since W_1 is a subspace, we have $cw_1 \in W_1$. Similarly $cw_2 \in W_2$. This shows cu is the sum of a vector in W_1 and a vector in W_2, hence $cu \in U$.

Suppose that $u, u' \in U$. Write $u = w_1 + w_2$ as above, and $u' = w_1' + w_2'$ for some $w_1' \in W_1$ and $w_2' \in W_2$. Then

$$u + u' = (w_1 + w_2) + (w_1' + w_2') = (w_1 + w_1') + (w_2 + w_2')$$

Since W_1 is a subspace, we have $w_1 + w_1' \in W_1$. Similarly $w_2 + w_2' \in W_2$. This shows $u + u'$ is the sum of a vector in W_1 and a vector in W_2, hence $u + u' \in U$.

We have shown that U is closed under scalar multiplication and under vector addition, hence it is a subspace of V.

(5) A square matrix A is called *antisymmetric* if $A^T = -A$. Show that the set of antisymmetric matrices is a subspace of $M_{3\times 3}(\mathbb{R})$ and find a basis for this subspace.

Solution. The transpose of a matrix satisfies $(A+B)^T = A^T + B^T$ and $(cA)^T = c(A^T)$. Hence if $A, B \in M_{3\times 3}(\mathbb{R})$ are antisymmetric, and $c \in \mathbb{R}$, we have

$$(A+B)^T = A^T + B^T = -A - B = -(A+B)$$

and

$$(cA)^T = c(A^T) = c(-A) = -(cA).$$

That is, $A+B$ and cA are also antisymmetric. This shows that the set of antisymmetric matrices is closed under scalar multiplication and under vector addition, hence it is a subspace of $M_{3\times 3}(\mathbb{R})$.

In what follows, let us denote the subspace of $M_{3\times 3}(\mathbb{R})$ consisting of antisymmetric matrices by W.

Let

$$S = \left\{ \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \right\}.$$

We claim that S is a basis for W. First, we show S is linearly independent: If we consider a linear combination of elements of S,

$$a \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + b \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} + c \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix}$$

then if this linear combination is equal to the zero matrix, by comparing the $(1,2)$, $(1,3)$, and $(2,3)$ entries we find $a = b = c = 0$. Thus S is linearly independent.

Next we show that S spans W. Consider a 3×3 matrix A, and denote its entries by

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}.$$

If $A \in W$ then $A^T = -A$ and we have

$$\begin{pmatrix} a & d & g \\ b & e & h \\ c & f & i \end{pmatrix} = \begin{pmatrix} -a & -b & -c \\ -d & -e & -f \\ -g & -h & -i \end{pmatrix}.$$

Comparing diagonal entries gives $a = -a$, $e = -e$, and $i = -i$. Hence $a = e = i = 0$. Comparing off-diagonal entries we find that $d = -b$, $g = -c$, and $h = -f$. Thus $A \in W$

can be written as

$$\begin{pmatrix} 0 & b & c \\ -b & 0 & f \\ -c & -f & 0 \end{pmatrix} = b\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + c\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} + f\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

which shows that A is a linear combination of elements of S. Hence S spans W.

We have shown S is a linearly independent set that generates W, therefore S is a basis of W.