Math 445 – David Dumas – Fall 2015

Midterm Exam Solutions

(1) (a) [2 points] Write the definition of *continuity* for a function $f : X \to Y$, where X and Y are topological spaces.

(b) [2 points] Write the definition of ε-δ *continuity* for a function $f : X \to Y$, where (X, d_X) and (Y, d_Y) are metric spaces.

(c) [6 points] Show that the topological and ε-δ definitions of continuity are equivalent for maps between metric spaces.

Solution. A function $f : X \to Y$ is called *continuous* if for every open set $V \subset Y$, the set $f^{-1}(V)$ is open.

A function $f : X \to Y$, where (X, d_X) and (Y, d_Y) are metric spaces, is ε-δ continuous if for every $\varepsilon > 0$ and every $x \in X$ there exists $\delta > 0$ such that $d_Y(f(x), f(x')) < \varepsilon$ for all $x' \in X$ with $d_X(x, x') < \delta$.

For a proof of the equivalence of these definitions when X and Y are metric spaces, see Theorem 21.1 in Munkres.

(2) [10 points] Show that a topological space is T_1 if and only if every finite set is closed.

(Recall that a topological space X is said to be T_1 if for every $x, y \in X$ with $x \neq y$ there exists an open set U such that $x \in U$ and $y \notin U$.)

Solution. Suppose X is T_1, and let $y \in X$. The set $X \setminus \{y\}$ is open because it contains a neighborhood of each of its points: For any $x \in X \setminus \{y\}$ the T_1 axiom gives a neighborhood U of x that does not contain y, or equivalently, so that $U \subset X \setminus \{y\}$.

Since $X \setminus \{y\}$ is open, the singleton set $\{y\}$ is closed. Finite unions of closed sets are closed, hence every finite subset of X is closed.

Conversely, suppose every finite set in X is closed. Given $x, y \in X$ with $x \neq y$, let $U = X \setminus \{y\}$. Since $\{y\}$ is closed, U is open. Since $x \in U$ and $y \notin U$, this shows that X is T_1.

(3) [10 points] Let X be a topological space and $\sim$ an equivalence relation on X. Suppose $X/{\sim}$ is connected and that each equivalence class of $\sim$ is connected. Show that X is connected.

Solution. Let $\pi : X \to X/{\sim}$ denote the quotient map associated to $\sim$. Suppose for contradiction that $X = A \cup B$ is a separation of X.

First suppose that there is an equivalence class C of $\sim$ that intersects both A and B non trivially. Then $C = (A \cap C) \cup (B \cap C)$ is a separation of C, which is connected by hypothesis, giving a contradiction.

If every equivalence class of $\sim$ is contained in either A or B, then $\pi(A)$ and $\pi(B)$ are disjoint and cover $X/\sim$. Furthermore $\pi(A)$ and $\pi(B)$ are open, since A and B are open and each is a union of equivalence classes. Thus $X/\sim = \pi(A) \cup \pi(B)$ is a separation of $X/\sim$, which is connected by hypothesis, giving a contradiction.

(4) Consider the set $\mathbb{R}$ with the finite-complement topology. (This is the topology in which the nonempty open sets are exactly the sets with finite complement.) Answer each of the following questions about this topological space, and give a proof of each answer.

(a) [$2\frac{1}{2}$ points] Is it Hausdorff?

(b) [$2\frac{1}{2}$ points] Is it connected?

(c) [$2\frac{1}{2}$ points] Is it path-connected?

(d) [$2\frac{1}{2}$ points] Is it compact?

Solution. Let X denote the topological space that is $\mathbb{R}$ equipped with the finite complement topology. We claim that X is connected, path-connected, and compact but not Hausdorff. That is, the answers are:

(a) No
(b) Yes
(c) Yes
(d) Yes

To prove this, first notice that $\mathbb{R}$ is infinite, so any two nonempty open sets in X have nontrivial intersection. Since the Hausdorff condition requires the existence of disjoint pairs of nonempty open sets, it follows that X is not Hausdorff.

To see that X is compact, consider an open cover $\mathscr{A}$. Choose $A_0 \in \mathscr{A}$, which is open and hence $\mathbb{R} \setminus A_0 = \{x_1, \ldots, x_n\}$ is finite. For each i with $1 \le i \le n$, let $A_i \in \mathscr{A}$ be a set from the cover such that $x_i \in A_i$. Then $\{A_0, A_1, \ldots, A_n\}$ is a finite subcover of $\mathscr{A}$.

Finally we consider (path-)connectedness. Let $a, b \in X$. Let $[a, b]$ denote the associated closed interval in $\mathbb{R}$ with the standard topology. The inclusion map $[a, b] \hookrightarrow X$ is continuous because the finite complement topology is coarser than the standard topology, and thus this inclusion gives a path from a to b in X. We have therefore shown that X is path-connected, and any path-connected space is connected.

(5) [10 points] Let X be an ordered set. Show that the order topology on X is Hausdorff.

Solution. Let $x, y \in X$, with $x \neq y$. Renaming the points if necessary, we can assume $x < y$. We seek disjoint neighborhoods of x and y.

If the open interval (x, y) is empty, then $U = (-\infty, y)$ and $V = (x, \infty)$ are disjoint, open in the order topology, $x \in U$, and $y \in V$.

Otherwise let $z \in (x, y)$. Then $U = (-\infty, z)$ and $V = (z, \infty)$ are disjoint, open in the order topology, $x \in U$, and $y \in V$.

Therefore, the order topology on X is Hausdorff.