**Math 445 – David Dumas – Fall 2015**

# Solution to Problem 1, Homework 4

(P1) Suppose $X$ is a set and $\mathscr{T}, \mathscr{T}'$ are topologies on $X$. In each part of this problem, determine whether or not the given condition implies that $\mathscr{T} = \mathscr{T}'$. If so, give a proof. If not, give a counterexample and prove that it is a counterexample.

(a) For every finite topological space $W$, a function $f : W \to X$ is continuous with respect to $\mathscr{T}$ if and only if it is continuous with respect to $\mathscr{T}'$.

**Solution:** This condition does **not** necessarily imply that $\mathscr{T} = \mathscr{T}'$.

For example, consider $\mathbb{R}$ with the standard topology ($\mathscr{T}$) and with the finite complement topology ($\mathscr{T}'$). Both of these topologies on $\mathbb{R}$ induce the discrete topology on any finite subspace of $\mathbb{R}$, and in particular they give the same subspace topology on $f(W)$ whenever $f : W \to \mathbb{R}$ is a function and $W$ is finite. Since the continuity of a function $f : W \to \mathbb{R}$ is equivalent to continuity of the associated function $f : W \to f(W)$, where $f(W) \subset \mathbb{R}$ is given the subspace topology, this shows that $f : W \to \mathbb{R}$, with $W$ finite, is continuous for $\mathscr{T}$ if and only if it is continuous for $\mathscr{T}'$.

(b) For every finite topological space $Y$, a function $f : X \to Y$ is continuous with respect to $\mathscr{T}$ if and only if it is continuous with respect to $\mathscr{T}'$.

**Solution:** This condition implies that $\mathscr{T} = \mathscr{T}'$. In fact, it suffices to consider a single space: If $Y = \{0, 1\}$ with the topology $\{\emptyset, \{1\}, \{0, 1\}\}$, then the indicator function $\chi_A : X \to Y$ of a subset $A \subset X$ is continuous if and only if $A$ is open.

(c) For every Hausdorff topological space $Y$, a function $f : X \to Y$ is continuous with respect to $\mathscr{T}$ if and only if it is continuous with respect to $\mathscr{T}'$.

**Solution:** This condition does **not** necessarily imply that $\mathscr{T} = \mathscr{T}'$.

For example, consider $X = \{0, 1\}$ with $\mathscr{T} = \{\emptyset, \{1\}, \{0, 1\}\}$ and $\mathscr{T}' = \{\emptyset, \{0\}, \{0, 1\}\}$. These topologies are distinct, but in both cases constant functions are the only continuous functions from $X$ to a Hausdorff topological space. To see this, suppose on the contrary that $f : X \to Y$ is continuous and not constant, with $Y$ Hausdorff. Choose $y, y' \in f(X)$ with $y \neq y'$. Then $y$ and $y'$ have disjoint open neighborhoods $U, U'$. Thus $f^{-1}(U)$ and $f^{-1}(U')$ are disjoint, open, and nonempty subset of $X$. But neither $\mathscr{T}$ nor $\mathscr{T}'$ has a pair of disjoint nonempty open sets, so this is a contradiction.