Math 445 – David Dumas – Fall 2015

Homework 9

Due Friday, November 6 at 1:00pm

Instructions:

- To receive full credit, a solution must be clear, concise, and correct.
- Problems marked with * are *extra credit problems* and these are *optional*.
- If a problem asks a question with a "yes" or "no" answer, you must provide a proof of whatever answer you give.

(—) From the textbook: 31.7, 31.9, 32.1, 32.9

(P1) * Let $\mathscr{T}$ denote the smallest topology on $\mathbb{Z}$ in which every arithmetic progression is an open set. (Recall an arithmetic progression is a set of the form $\{ak+b \mid k \in \mathbb{Z}\}$ where $a,b \in \mathbb{Z}$ and $a \neq 0$.)

(a) Show that every arithmetic progression is a clopen set with respect to $\mathscr{T}$.

(b) Show that all of the nonempty open sets in $\mathscr{T}$ are infinite.

(c) Let $P = \{2,3,5,\ldots\}$ be the set of prime numbers, and for each $p \in P$ let D_p denote the set of integers divisible by p. Show that

$$\bigcup_{p\in P} D_p = \mathbb{Z} \setminus \{-1,1\}.$$

Conclude that there are infinitely many prime numbers, for otherwise $\{-1,1\}$ would open, contradicting (b).