Math 445 – David Dumas – Fall 2015

Homework 12

Due Friday, December 4 at 1:00pm

Instructions:

- To receive full credit, a solution must be clear, concise, and correct.
- Problems marked with * are *extra credit problems* and these are *optional*.

(—) From the textbook: 46.7, 45.8 (longer problem; 20 points)

In the problems below, (X,d) is a metric space and $d_H(A,B)$ denotes the Hausdorff distance between subsets $A,B \subset X$, which is defined as

$$d_H(A,B) = \inf\{\varepsilon \mid A \subset N_\varepsilon(B) \text{ and } B \subset N_\varepsilon(A)\}.$$

We denote by $\mathscr{B}(X)$ the collection of all closed, bounded subsets of X. Recall that $(\mathscr{B}(X), d_H)$ is a metric space.

(P1) Show that for any subset $A \subset X$, we have $d_H(A,\bar{A}) = 0$.

(P2) Show that if X is compact, then for every ε there exists a *finite* subset $A(\varepsilon) \subset X$ such that $d_H(X, A(\varepsilon)) < \varepsilon$. Give an example to show that the converse does not hold (i.e., there are non-compact metric spaces with this property).

(P3) * Suppose X is a proper metric space†, that A_n is a sequence in $\mathscr{B}(X)$, and that $A_n \supset A_{n+1}$ for all n. Show that A_n converges (in the topology induced by d_H) and that the limit is $\cap_n A_n$.

(P4) * Let $\mathscr{C}(\mathbb{R}^n)$ denote the collection of all *convex* subsets of $\mathbb{R}^n$ that are also closed and bounded. (Recall that a set $\Omega \subset \mathbb{R}^n$ is convex if for every $a,b \in \Omega$, the line segment $\overline{ab}$ is contained in Ω.) Show that $\mathscr{C}(\mathbb{R}^n)$ is a closed subset of $\mathscr{B}(\mathbb{R}^n)$ in the topology induced by d_H.

(P5) * Suppose X is a proper metric space† and consider the subset of $\mathscr{B}(X)$ consisting of the closed, bounded sets which are also *connected*. Is this a closed set in the topology induced by d_H?

(P6) * Show that if $U \subset \mathbb{R}^n$ is open and bounded, then $\bar{U}$ is the limit (with respect to d_H) of the sequence of finite sets U_k defined by

$$U_k = U \cap \left(\frac{1}{k}\mathbb{Z}^n\right)$$

where $\frac{1}{k}\mathbb{Z}^n$ denotes the image of $\mathbb{Z}^n$ under the map $(x_1,\ldots,x_n) \mapsto (\frac{1}{k}x_1,\ldots,\frac{1}{k}x_n)$.

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†An earlier version of this assignment omitted the properness assumption in these two problems