Math 445 – David Dumas – Fall 2015

Homework 10

Due Monday, November 16 at 1:00pm

Instructions:

- To receive full credit, a solution must be clear, concise, and correct.
- Unlike the other recent assignments, this one does **not** have any optional problems.

(—) From the textbook: 33.3, 34.3

(P1) Prove "Fact 4" from Clark's notes on Tychonoff's theorem, i.e.:

Let $X = \prod_{\alpha \in J} X_\alpha$ be a product of topological spaces, with the product topology, and denote by $\pi_\alpha : X \to X_\alpha$ the projection onto a factor. Let $\mathscr{F}$ be a filter on X and $p \in X$. Show $\mathscr{F} \to p$ if and only if for all $\alpha \in J$ we have $\pi_\alpha(\mathscr{F}) \to \pi_\alpha(p)$.

(Even if we sketched a proof in lecture, you should fill in the details.)

(P2) Let X be an infinite set. For each $p \in X$ there is a filter

$$\mu_p = \{A \subset X \mid p \in A\}$$

Any filter of this form is called *principal*. Another filter on X is the *finite complement filter*,

$$\mu_\infty = \{A \subset X \mid X \setminus A \text{ is finite}\}.$$

(a) Show that every principal filter μ_p is an ultrafilter.
(b) Show that μ_∞ is not an ultrafilter.
(c) Show that every non-principal ultrafilter contains μ_∞.

(P3) Show that a topological space X is Hausdorff if and only if every ultrafilter $\mathscr{F}$ on X converges to at most one point.

(P4) Let ω be a non-principal ultrafilter on $\mathbb{N}$. Let $(x_1, x_2, \ldots)$ be a bounded sequence of real numbers.

(a) Show that there exists a unique real number a such that for every $\varepsilon > 0$,

$$\{n \mid |a - x_n| < \varepsilon\} \in \omega.$$

(b) The real number a from (a) is called the ω-limit of the sequence x_n, denoted

$$a = \lim_\omega x_n.$$

Show that if the sequence x_n converges in the usual sense, then it converges to its ω-limit. That is,

$$\lim_{n \to \infty} x_n = \lim_\omega x_n$$

if the left-hand side exists.

(c) Show that

$$\liminf_{n\to\infty} x_n \leqslant \lim_{\omega} x_n \leqslant \limsup_{n\to\infty} x_n.$$

(d) Show that if y_n is another bounded sequence, and if $c \in \mathbb{R}$, then

$$\lim_{\omega} cx_n = c \lim_{\omega} x_n$$

and

$$\lim_{\omega}(x_n + y_n) = \lim_{\omega} x_n + \lim_{\omega} y_n.$$

This problem shows that the choice of a non-principal ultrafilter on $\mathbb{N}$ gives a way to consistently assign a "limit" to every bounded sequence of real numbers.