Math 550 / David Dumas / Fall 2014

Problems

Please note:

- This list was last updated on November 30, 2014.
- Problems marked with * are *challenge problems*.
- Some problems are adapted from the course texts; these are marked with a citation of the form [*Author*, *Chapter.Problem*] or similar.
- Sometimes the "answer" to one of these problems might come up in lecture. *You can still complete such problems for credit*, but you must carefully write up all of the details.

(1) * Prove Elie Cartan's formula $\mathscr{L}_X \omega = i_X(d\omega) + d(i_X\omega)$ by the following strategy:
  (a) Suppose $T : \Omega^*(M) \to \Omega^*(M)$ is linear map (preserving degree of forms, $T : \Omega^k \to \Omega^k$) satisfying
    (i) $T(\alpha \wedge \beta) = (T\alpha) \wedge \beta + \alpha \wedge (T\beta)$, and
    (ii) $T(df) = d(Tf)$ for all $f \in \Omega^0(M) = C^\infty(M)$.
  Show that T is uniquely determined by its action on $\Omega^0(M)$.
  (b) Show that each side of Cartan's formula (i.e. the maps $\omega \mapsto \mathscr{L}_X\omega$ and $\omega \mapsto i_X(d\omega) + d(i_X\omega)$) satisfies the properties (i) and (ii) above.
  (c) Show that Cartan's formula holds for functions, and in fact that

$$\mathscr{L}_X f = (i_X(df) + d(i_X f)) = Xf.$$

(2) * Prove that, as mentioned in class, the "radial flow" approach to the Poincaré lemma can be generalized to show that no compact orientable manifold is smoothly contractible to a point. In other words:
  (a) Adapt the radial flow method to show that a manifold smoothly contractible to a point has no de Rham cohomology in positive degrees. (This should involve a homotopy operator that averages pullbacks of a form over the family of maps that give the smooth contraction.)
  (b) Show $H^n(M) \neq 0$ for M a compact orientable manifold.

(3) Show how the Poincaré lemma implies the following facts from vector calculus, and work out the formula that it gives for a primitive in each case:
  (a) Every smooth vector field on $\mathbb{R}^3$ with zero curl is the gradient of a function.
  (b) Every smooth vector field on $\mathbb{R}^3$ with zero divergence is the curl of a vector field.
  (c) Every smooth function on $\mathbb{R}^3$ is the divergence of a vector field.

(4) (Lee, 11-1) Show that the wedge product of forms induces a well-defined bilinear map $H^p(M) \times H^q(M) \to H^{p+q}(M)$.

(5) (Lee, 11-6) Let M be a compact, connected, orientable n-manifold and $p \in M$ a point. Decompose $M = U \cup V$ where $U \simeq \mathbb{R}^n$ is an open neighborhood of p and $V = M \setminus \{p\}$. Show that the connecting homomorphism

$$\delta : H^{n-1}(U \cap V) \to H^n(M)$$

of the Mayer-Vietoris sequence for de Rham cohomology is an isomorphism. Hint: $U \cap V$ is diffeomorphic to $\mathbb{R}^n \setminus \{0\}$; construct an explicit $(n-1)$-form on this set which is closed but not exact.

(6) (Spivak, 7.4) Show that the interior product $i_X : \Omega^*(M) \to \Omega^{*-1}(M)$ satisfies
 (a) $i_X(i_Y \omega) = -i_Y(i_X \omega)$
 (b) $i_X(\alpha \wedge \beta) = (i_X \alpha) \wedge \beta + (-1)^k \alpha \wedge (i_X \beta)$ where $\alpha \in \Omega^k(M)$

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(7) (Spivak, 7.6)
 (a) Show that any alternating 2-form on a 3-dimensional vector space is the wedge product of two 1-forms.
 (b) Give an example of a 2-form on a 4-dimensional vector space that is not the wedge product of two 1-forms.

(8) * (Spivak, 7.25) Show that any star-shaped open set $U \subset \mathbb{R}^n$ is diffeomorphic to $\mathbb{R}^n$.
 - Hint: This is more complicated than it looks because the distance from the origin to the boundary of U can vary discontinuously. For example, take the open unit ball in $\mathbb{R}^3$ and remove the line segment from $(1/2,0,0)$ to $(1,0,0)$.
 - Disclosure: This is really a problem in differential topology, and as such it is not closely related to any of the techniques we cover in this course (except perhaps the Whitney Approximation Theorem). However, it's a nice problem.

(9) Let $M \subset \mathbb{R}^n$ be a compact embedded submanifold. Define a function $\mathbb{R}^n \to \mathbb{R}^{\geq 0}$ by

$$d_M(x) = \inf_{y \in M} |x - y|,$$

where $|x|$ is the usual norm on $\mathbb{R}^n$. For any $\varepsilon > 0$ let $U_\varepsilon = d_M^{-1}([0,\varepsilon))$. Show that there exists $\varepsilon > 0$ such that
 - The restriction of d_M to $U_\varepsilon \setminus M$ is a smooth function,
 - For each $x \in U_\varepsilon$ there exists a unique point $r(x) \in M$ with $|x - r(x)| = d_M(x)$, and
 - The function $r : U_\varepsilon \to M$ thus defined is a smooth retraction.

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(10) Let M be a smooth manifold and α a closed 1-form. Define the *α-deformed exterior derivative* $d_\alpha : \Omega^k(M) \to \Omega^{k+1}(M)$ as follows:

$$d_\alpha(\omega) := d\omega + \alpha \wedge \omega$$

 (a) Show that $(\Omega^*(M), d_\alpha)$ is a cochain complex (the *deformed de Rham complex*).
 (b) Show that if α is exact, then the cohomology groups $H^*_\alpha(M)$ of the deformed de Rham complex are isomorphic to the usual de Rham cohomology groups $H^*(M)$.
 (c) * Give an example of M and α such that $H^*(M) \not\cong H^*_\alpha(M)$. (This part of the problem is optional, but completing it makes this count as a challenge problem.)

(11) (a) Let D be a finite set of points in $\mathbb{R}^2$. Compute the de Rham cohomology $H^*(\mathbb{R}^2 \setminus D)$ by exhibiting an explicit basis in each degree. (You must show that your forms are a basis by verifying that they are closed, linearly independent modulo exact forms, and that they span the cohomology group.)
 (b) Do the same thing for the complement of a finite set in $\mathbb{R}^3$.

(12) The de Rham theorem shows that the de Rham cohomology $H^*(M)$ and the singular homology $H_*(M,\mathbb{R})$ of a smooth manifold M are dual vector spaces. For each of the manifolds below, give *dual bases* of de Rham cohomology and singular homology in each degree:
   (a) The 2-torus, $\mathbb{R}^2/\mathbb{Z}^2$.
   (b) The 2-sphere
   (c) The 3-sphere
   Your basis for each de Rham space should consist of explicit differential forms. For the singular spaces the basis elements can be explicit chains or the fundamental classes of (explicitly described) embedded oriented submanifolds.

(13) * Same question as above, but for these manifolds:
   (a) The complex projective plane, $\mathbb{CP}^2$.
   (b) An oriented smooth surface of genus 2.
   (c) The complement of three lines in $\mathbb{R}^3$, each pair of which meet in a single point (but where no point lies on all three lines).

(14) (Lee, 18-6) Let $H_c^*(M)$ denote the cohomology of the chain complex $\Omega_c^\star(M)$ of *compactly supported* differential forms (with the usual exterior derivative as differential).
   (a) Give an example to show that, in general, a smooth map $F: M \to N$ between manifolds does *not* induce a pullback map $F^*: H_c^\star(N) \to H_c^\star(M)$.
   (b) Show that an inclusion of an open set $i: U \hookrightarrow M$ induces an "extension by zero" map

$$i_*: H_c^\star(U) \to H_c^\star(M)$$

   and hence that, in this sense, compactly supported cohomology exhibits *covariant* behavior.

(15) (Lee, 18-6) Show that if $\{U,V\}$ is an open cover of M, then there is a long exact sequence of compactly supported cohomology groups

$$\cdots \to H_c^k(U\cap V) \to H_c^k(U)\oplus H_c^k(V) \to H_c^k(M) \to H_c^{k+1}(U\cap V) \to \cdots$$

   Give an explicit description of the connecting homomorphism and a formula for the maps between cohomology groups of equal degrees in terms of the inclusions among $U\cap V, U, V, M$.

(16) Show that the de Rham map is an isomorphism of rings, taking the wedge product on $H_{dR}^*(M)$ (see Problem 4) to the cup product on $H^*(M,\mathbb{R})$.

(17) * (Lee, 18-7) The Poincaré duality theorem for de Rham cohomology asserts that if M is an oriented smooth n-manifold, then there is a nondegenerate bilinear form

$$P: H^k(M)\times H_c^{n-k}(M) \to \mathbb{R}$$

   given by

$$P([\omega],[\eta]) = \int_M \omega\wedge\eta.$$

   Equivalently this integration map gives an isomorphism $H^k(M) \simeq H_c^{n-k}(M)^*$, and for compact M this implies $H^k(M)\simeq H^{n-k}(M)^*$.
   (a) Show that P is well-defined on cohomology.
   (b) Imitate the proof of the de Rham theorem given in lecture to prove the Poincaré duality theorem. You will need to use a version of the Poincaré lemma for compactly supported forms (See e.g. Lemma 17.27 in Lee).

(18) * (Bott-Tu, 5.16) Let M be an oriented manifold of dimension n. Let $S \subset M$ be an embedded oriented submanifold of dimension k.

(a) Suppose S is closed (as a subset of M). Integrating compactly supported k-forms on M over S gives a linear functional $H^k_c(M) \to \mathbb{R}$. By Poincaré duality, this functional is represented by a class $[\eta_S] \in H^{n-k}(M)$, the *closed Poincaré dual* of S. Describe this class for each of these cases:

(i) $M = \mathbb{R}^n$, $S =$point.

(ii) $M = \mathbb{R}^2 \setminus \{0\}$, $S =$ unit circle.

(iii) $M = \mathbb{R}^2 \setminus \{0\}$, $S =$ ray. (That is, $S = \{(x,0) \mid x > 0\}$.)

(b) Suppose S is compact. Integrating arbitrary k-forms on M over S gives a linear functional $H^k(M) \to \mathbb{R}$. By Poincaré duality this functional is represented by a class $[\eta'_S] \in H^{n-k}_c(M)$, the *compact Poincaré dual* of S. Describe this class for the submanifolds (i) and (ii) from part (a).

Remark: Notice that these two notions of Poincaré dual agree if M is compact.

(19) Let M be a compact oriented manifold and S a compact oriented embedded submanifold. Let $[\eta_s] \in H^{n-k}(M)$ be the Poincaré dual of S, in the sense of the previous problem. Show that for any open neighborhood U of S in M, the class $[\eta_s]$ can be represented by a form that is supported inside U.

(20) Show that these two definitions of a vector field on a smooth manifold are equivalent, and describe the equivalence.

Def 1: A vector field is a derivation $X : C^\infty(M) \to C^\infty(M)$, i.e. a $\mathbb{R}$-linear map satisfying $X(fg) = X(f)g + fX(g)$.

Def 2: A vector field is a smooth map $X : M \to TM$ such that $\pi \circ X = \mathrm{id}_M$, where $\pi : TM \to M$ is the projection taking a tangent vector to its base point.

(21) The real quaternions are the 4-dimensional $\mathbb{R}$-algebra spanned (as a vector space) by $\{1, i, j, ij\}$ subject to the relations $ij = -ji$, $i^2 = j^2 = -1$. The norm of a quaternion w is $\sqrt{w\bar{w}}$ where if $w = a + bi + cj + dij$ for $a, b, c, d \in \mathbb{R}$ then $\bar{w} = a - bi - cj - dij$. (Note that $w\bar{w}$ is always real and nonnegative.)

(a) Show that the unit quaternions form a Lie group that is diffeomorphic to S^3 (the 3-sphere).

(b) Show that the Lie algebra of this Lie group is naturally isomorphic to $\mathbb{R}^3$ equipped with the vector cross product as its Lie bracket.

(c) Compute the Maurer-Cartan form of this Lie group in terms of the differentials of the coordinate functions a, b, c, d.

(22) (a) Show that S^2 cannot be given the structure of a Lie group.

(b) Show that $\mathbb{RP}^3$ has the structure of a Lie group.

(c) Give an example of two *connected* Lie groups that are diffeomorphic but not isomorphic as Lie groups.

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(23) * Show that S^7 cannot be given the structure of a Lie group.

(24) Classify Lie algebras of dimension at most 2 over $\mathbb{R}$ up to isomorphism.

(25) * Classify Lie algebras of dimension at most 3 over $\mathbb{C}$ up to isomorphism.

(26) * Classify Lie algebras of dimension at most 3 over $\mathbb{R}$ up to isomorphism.

(27) Show that a left-invariant vector field on a Lie group is complete, i.e. the resulting flow Φ_t is defined on all of $G \times \mathbb{R}$.

(28) Fill in the details of this proposition whose proof was sketched in lecture: If a differential form $\omega \in \Omega^k(G)$ is left-invariant and if $\omega(e) \in (\mathfrak{g}^*)^{\otimes k}$ is invariant under the representation $(\mathrm{ad}^*)^{\otimes k}$ of $\mathfrak{g}$, then ω is also right-invariant.

(29) Let H_{2n+1} denote the Lie group structure on $\mathbb{R}^{2n+1}$ given by $(a,b,c)\cdot(a',b',c') = (a+a', b+b', c+c'+\langle a,b'\rangle)$ where $a,a',b,b' \in R^n$ and $c,c' \in \mathbb{R}$. Here $\langle \cdot,\cdot\rangle$ denotes the standard inner product on $\mathbb{R}^n$.
  (a) Describe the Lie algebra $\mathfrak{h}_{2n+1} = \mathrm{Lie}(H_{2n+1})$ explicitly (for example by listing all nontrivial brackets for some basis)
  (b) Compute the exponential map $\exp : \mathfrak{h}_{2n+1} \to H_{2n+1}$.

(30) * Compute the Lie algebra cohomology $H^*(\mathfrak{g})$ for $\mathfrak{g} = \mathfrak{h}_3$ and $\mathfrak{g} = \mathfrak{h}_5$. (See the previous problem for the definition of these Lie algebras.)

(31) Show that the exponential map for $\mathrm{SL}(2,\mathbb{R})$ is not surjective.

(32) For each of the elements of $\mathfrak{sl}(2,\mathbb{R})$ below, compute the corresponding 1-parameter subgroup of $\mathrm{SL}(2,\mathbb{R})$. Then describe the orbits of points in $\mathbb{H}$ under the images of these subgroups in $\mathrm{PSL}(2,\mathbb{R})$. (Here $\mathbb{H}$ is the upper half-plane in $\mathbb{C}$ on which $\mathrm{PSL}(2,\mathbb{R})$ acts by Möbius transformations, $z \mapsto \frac{az+b}{cz+d}$).
  (a) $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$
  (b) $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$
  (c) $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$
  (d) $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

(33) Show that the torus $T^2 = S^1 \times S^1$ is the only compact, connected, 2-dimensional Lie group. (Hint: Problem 24 and $\chi = 0$ will be helpful.)

(34) * (This problem assumes some familiarity with Riemannian geometry.) Suppose $b(\cdot,\cdot)$ is an ad-invariant positive definite inner product on $\mathfrak{g} = \mathrm{Lie}(G)$. Using the trivialization of the tangent bundle, $TG \simeq G \times \mathfrak{g}$ we can consider b as a Riemannian metric on G. Show that 1-parameter subgroups of G are geodesics of this metric.

(35) (a) Show that $\mathrm{SL}(n,\mathbb{R})$ is diffeomorphic to $\mathrm{SO}(n) \times \mathbb{R}^d$ for a suitable positive integer d.
  (b) Show that the diffeomorphism in part (a) cannot be a Lie group isomorphism (regardless of what Lie group structure is applied to $\mathbb{R}^d$).

(36) Let $\omega \in \Omega^3(\mathrm{SU}(2))$ be the bi-invariant 3-form associated to the ad-invariant Lie algebra cocycle

$$\omega_e(x,y,z) = B([x,y],z)$$

where B is the Killing form on $\mathfrak{su}(2)$, i.e. $B(x,y) = \mathrm{tr}(\mathrm{ad}_x \circ \mathrm{ad}_y)$.

(a) Show that ω is proportional to the standard volume form when $\mathrm{SU}(2) \simeq S^3$ is considered as the unit sphere in $\mathbb{R}^4$. (One definition of the "standard volume form" on S^3 is the pullback via $S^3 \hookrightarrow \mathbb{R}^4$ of the 3-form $i_\rho(dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4)$ where ρ is the unit vector field on $\mathbb{R}^4$ pointing away from the origin.)

(b) Calculate the coefficient of proportionality relating these volume forms.

(37) Give an example of a Lie algebra $\mathfrak{g}$ that has a non-split central extension $\tilde{\mathfrak{g}}$ by $\mathbb{R}$. (In particular, prove that your extension is not isomorphic to the Lie algebra $\mathfrak{g} \oplus \mathbb{R}$.) Also exhibit the corresponding element $\omega \in H^2(\mathfrak{g})$.

(38) Suppose G is a Lie group with Lie algebra $\mathfrak{g}$, and $E \subset \mathfrak{g}$ is a subspace. Let ω_G denote the Maurer-Cartan form on G, and let $p : \mathfrak{g} \to \mathfrak{g}/E$ denote the quotient map. Then $p \circ \omega_G$ is a 1-form on G with values in $\mathfrak{g}/E$.

(a) What conditions must the subspace E satisfy in order for the ideal generated by $p \circ \omega_G$ in $\Omega^*(G)$ to be differential?

(b) In cases where this ideal is differential, describe the leaf L_e of the associated foliation of G.

(As in lecture, we use the convention that "the ideal generated by a V-valued 1-form α" is shorthand for the ideal in Ω^* generated by all 1-forms obtained by composing α with a linear functional $V \to \mathbb{R}$.)

(39) (a) Show that the pullback of a fiber bundle, as defined in lecture, is a fiber bundle. That is, if $\pi : E \to B$ is a fiber bundle with fiber F, and if $\phi : B' \to B$ is a continuous map, then defining

$$\phi^* E = \{(e, b') \mid e \in E, b' \in B, \pi(e) = \phi(b')\}$$

gives a fiber bundle with projection $\pi' : \phi^* E \to B'$ given by $(e, b') \mapsto b'$.

(b) Show that the product of fiber bundles over a fixed base, as defined in lecture, is a fiber bundle. Here $E \times_B E'$ is defined as $\{(e, e') \in E \times E' \mid \pi(e) = \pi'(e')\}$ with projection $(e, e') \mapsto \pi(e)$.

(40) Give an example of a non-trivial fiber bundle $\pi : E \to B$ and a fiber bundle $\pi' : E' \to B$ such that $E \times_B E'$ is trivial.

(41) Give an example of a topological space E that is the total space of infinitely many non-isomorphic fiber bundles. (You need to describe E, construct an infinite collection of bundles $\pi_n : E \to B_n$, and then show that no two of the bundles are isomorphic.)

(42) * Give an example of a compact, connected manifold E that is the total space of infinitely many non-isomorphic smooth fiber bundles, each of which has connected fibers.

(43) Show that the orthogonal frame bundle of a Riemannian manifold is a principal bundle. (This amounts to filling in the details about constructing local trivializations.)

(44) Suppose that X is a topological space and $\phi : X \to X$ is a homeomorphism. Define

$$E = (X \times [0,1]) / \sim$$

where $(\phi(x), 1) \sim (x, 0)$. Show that E has a the structure of a fiber bundle over S^1 with fiber X.

(45) Give an example of a fiber bundle (with fiber F) that does not admit a G-structure for any finite group $G \subset \mathrm{Homeo}(F)$.

(46) Let $\pi : P \to B$ be a principal G-bundle and F a left G-space. Prove that the two definitions of the F-bundle associated to P are equivalent, and that each gives a G-structure on a fiber bundle over B with typical fiber F. The definitions are:

- Global quotient definition: The group G has a left action on the product space $P \times F$ by $g \cdot (p, f) = (p \cdot g^{-1}, g \cdot f)$. Let $P \times_G F := (P \times F)/G$ denote the quotient space, and $\tilde{\pi} : P \times_G F \to B$ the map induced by $(p, f) \mapsto \pi(p)$. Then $(P \times_G F, B, \tilde{\pi}, F)$ is the associated fiber bundle.
- Local trivialization definition: Let $\{U_\alpha\}_{\alpha \in A}$ be an open cover of B with associated local trivializations of P given by $\phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times G$, and fiber transition maps $t_{\alpha\beta} : U_\alpha \cap U_\beta \to G$. Define $P \times_G F$ as the quotient space of

$$\bigsqcup_{\alpha \in A} (U_\alpha \times F)$$

by the equivalence relation generated by

$$(x, f) \in (U_\beta \times F) \;\sim\; (x, t_{\alpha\beta}(x) \cdot f) \in (U_\alpha \times F)$$

where $x \in U_\alpha \cap U_\beta$. The projection map $\tilde{\pi} : P \times_G F \to B$ is induced by the maps $(x, f) \mapsto x$ on each $U_\alpha \times F$ component.

(47) (a) Classify the principal $\mathbb{Z}/2$ bundles over S^1 up to isomorphism.
(b) Consider the $\mathbb{Z}/2$ action on $S^1 = \mathbb{R}/\mathbb{Z}$ where the nontrivial element acts as the *flip* $x \mapsto -x$. For each principal $\mathbb{Z}/2$-bundle over S^1, describe the topology of the associated S^1 fiber bundle over S^1.
(c) Consider the $\mathbb{Z}/2$ action on $S^1 = \mathbb{R}/\mathbb{Z}$ where the nontrivial element acts as the half-turn $x \mapsto x + \frac{1}{2}$. For each principal $\mathbb{Z}/2$-bundle over S^1, describe the topology of the associated S^1 fiber bundle over S^1.

(48) Recall that S^3 can be identified the total space of a principal S^1-bundle over S^2. Thus for each smooth action of S^1 on a manifold F, we have an associated fiber bundle $S^3 \times_{S^1} F$.

For each integer n, the Lie group S^1 acts on itself by multiplication by z^n. Call this left S^1-space F_n. What is the total space of the resulting 3-manifold $E_n := S^3 \times_{S^1} F_n$? (That is, identify it as a "manifold you know" from another construction.)

Example: $E_0 = S^2 \times S^1$, and $E_1 = S^3$.

(49) What is the relation between these two notions?

- A smooth real vector bundle of rank n over a manifold M
- A smooth principal $\mathbb{R}^n$-bundle over M (where $\mathbb{R}^n$ is considered as an abelian Lie group)

Prove your claims, e.g. by showing they are equivalent notions, or that one is more restrictive, or that each one includes some objects that do not belong to the other class.

(50) Prove that these two definitions of a *torsion-free connection* on the frame bundle FM are equivalent.

Def 1: Let $\omega \in \Omega^1(FM, \mathfrak{gl}(n, \mathbb{R}))$ be a connection on the frame bundle, and let $\Theta \in \Omega^1(FM, \mathbb{R}^n)$ be the solder form, i.e.

$$\Theta_{(x,\phi)}(\xi) = \phi^{-1}(d\pi(\xi)).$$

Then we say ω is torsion-free if

$$d\Theta + \omega \wedge \theta = 0$$

where $\omega \wedge \theta \in \Omega^2(FM, \mathbb{R}^n)$ is obtained by taking the wedge product of forms (resulting in an element of $\Omega^2(FM, \mathfrak{gl}(n,\mathbb{R}) \otimes \mathbb{R}^n)$), and then composing this with the linear action map $\mathfrak{gl}(n,\mathbb{R}) \otimes \mathbb{R}^n \to \mathbb{R}^n$.

Def 2: First observe that any local frame $(\xi_1, \ldots, \xi_n)$ with $\xi_i \in \mathrm{Vect}(U)$, $U \subset M$, determines a local section $\sigma : U \to FM$ by

$$\sigma(x) = (x, \phi_x), \;\; \phi_x(e_i) = \xi_i(x).$$

Now let H be the horizontal distribution of a connection on the frame bundle. We say the connection is torsion-free if for each $p \in FM$ the horizontal space H_p is the tangent space of a local section σ that arises from a Lie-commuting local frame near $\pi(p)$.

(51) Show that for any smooth manifold M, the total space of the frame bundle FM is a parallelizable manifold.

(52) (Cannas da Silva, 2.1) Let V be a vector space over $\mathbb{R}$ of dimension $2n$. Show that an element $\alpha \in \bigwedge^2(V^*)$ is symplectic (that is, (V,α) is a symplectic vector space) if and only if α^n is nonzero. Here $\alpha^n = \alpha \wedge \cdots \wedge \alpha$.

(53) Give an example of a compact connected orientable manifold M of even dimension which has $H^{2k}(M,\mathbb{R}) \neq 0$ for each k, but which does not admit a symplectic structure.

(54) Let (V,ω) be a symplectic vector space.
  (a) Show that the set of Lagrangian subspaces of V, considered as a subset of the Grassmannian, is a compact connected smooth manifold.
  (b) Show that the set of symplectic subspaces of V of a fixed dimension, considered as a subset of the Grassmannian, is a connected smooth manifold. Show that except in dimensions 0 and $\dim(V)$, this manifold is not compact.

(55) * An *almost symplectic structure* on a manifold M is a nondegenerate alternating 2-form $\omega \in \Omega^2(M)$. Give an example of a manifold that has an almost symplectic structure but no symplectic structure.

(56) * A symplectic structure on a manifold M of dimension $2n$ gives a reduction $F_{\mathrm{Sp}}M \subset FM$ of the frame bundle to the symplectic group $\mathrm{Sp}(2n,\mathbb{R}) \subset \mathrm{GL}(2n,\mathbb{R})$. Show that a reduction $F_{\mathrm{Sp}}M$ arises from a symplectic structure ω if and only if $F_{\mathrm{Sp}}M$ has a torsion-free connection.

(57) Let $\mathfrak{g}$ be a finite-dimensional Lie algebra, and $\mathfrak{g}^*$ its dual vector space. Consider $\mathfrak{g}^* \simeq \mathbb{R}^n$ as a smooth manifold. We have an embedding $e : \mathfrak{g} \to C^\infty(\mathfrak{g}^*)$ by $e(x)(y) = y(x)$. Show that $C^\infty(\mathfrak{g}^*)$ has a unique Poisson algebra structure such that $\{e(x), e(y)\} = e([x,y])$.

(58) * Show that the Poisson structure constructed in the previous problem does *not* arise from any symplectic structure on the manifold $\mathfrak{g}^*$.

(59) * (This problem assumes some familiarity with Riemannian geometry.) Show that, as claimed in lecture, the geodesic equation on a Riemannian manifold (M,g) corresponds to the Hamiltonian

flow of the function $H(x,v) = \frac{1}{2}g(v,v)$ on TM, where TM has the symplectic structure arising from the isomorphism $TM \simeq T^*M$ given by g.