Math 180 / David Dumas / Fall 2014

Solution and Rubric for Quiz 10 (Fri Oct 31)

Problem: Give an example of a function with exactly two critical points.

The problem is very open-ended, and offers many possible routes to a solution. Here are three.

Solution 1: Many quadratic polynomials have two zeros, so one way to get a function with two critical points would be to have $f'(x)$ be a quadratic polynomial. A cubic polynomial $f(x)$ would have this property. Trying a random cubic polynomial is likely to work, but we illustrate here how a few failed attempts might guide one to a right answer.

We could try

$$f(x) = x^3 + 1$$

but then

$$f'(x) = 3x^2$$

has only one zero. Similarly $f(x) = x^3 + \text{constant}$ will not work.

Next we might try adding an x term,

$$f(x) = x^3 + x$$

but then

$$f'(x) = 3x^2 + 1$$

and this function has no zeros at all. But if we change the sign of x, letting

$$\boxed{f(x) = x^3 - x}$$

then

$$f'(x) = 3x^2 - 1$$

so f has two critical points (at $x = \pm 1/\sqrt{3}$), as required. Another option would be to add an x^2 term with any sign, e.g.

$$\boxed{f(x) = x^3 + x^2}$$

so that

$$f'(x) = 3x^2 + 2x = x(3x+2)$$

and there are critical points at $x = 0$ and $x = -2/3$.

Solution 2: The problem did not specify a domain the function is supposed to be defined on, so we can take a function with many critical points and restrict its domain to an interval that only contains two of them.

The function $\sin(x)$ oscillates, with one maximum and one minimum in each interval of width 2π. So if we restrict it to one of these intervals, we get exactly two critical points:

$$\boxed{f(x) = \sin(x) \text{ on the interval } [0, 2\pi]}$$

Solution 3: We can write a piecewise-defined function, ensuring that the graph has a corner at two points, and that there are no other critical points. One way to ensure no additional critical points exist is to use linear functions in the definition, which have constant derivative.

We can take, for example,

$$\boxed{f(x) = \begin{cases} x & \text{if } x \le 0 \\ -x & \text{if } 0 < x \le 1 \\ x-1 & \text{if } 1 < x \end{cases}}$$

In fact, the problem did not say that the function is continuous, so we could even make a definition that has critical points because of discontinuities, e.g.

$$\boxed{f(x) = \begin{cases} x & \text{if } x \le 0 \\ x+1 & \text{if } 0 < x \le 1 \\ x+2 & \text{if } 1 < x \end{cases}}$$

However, a more carefully-worded version of this problem would probably require the answer to be continuous, as it is unusual to study the critical points of discontinuous functions.

Rubric:

- If the final answer is correct, and is supported by clear and correct work: $\boxed{\text{4 points}}$
- Otherwise, if the final answer is a function with at least one critical point: $\boxed{\text{3 points}}$
- Otherwise, if a clear answer is given which is a function: $\boxed{\text{1 point}}$