Math 180 Written Homework

Assignment #9

Due **Tuesday, November 18th** at the beginning of your discussion class.

Directions. You are welcome to work on the following problems with other MATH 180 students, but your solutions must be hand-written, by your own hand, representing your understanding of the material. Word-by-word copying from another student or any other source is unacceptable. Any work without the proper justification will receive no credit. The list of problem solutions is to be submitted to your TA at the beginning of the discussion class listed above. No late homework will be accepted.

1. Compute the limit $\lim_{x\to 0^+} x^{\sin x}$.

   Solution: We may write:

   $$x^{\sin x} = e^{\ln\left(x^{\sin x}\right)} = e^{\sin x \ln x}$$

   Since the exponential function is a continuous function, it suffices to find the limit:

   $$\lim_{x\to 0^+} \sin x \ln x$$

   If it exists, then we will have:

   $$\lim_{x\to 0^+} e^{\sin x \ln x} = e^{\lim_{x\to 0^+} \sin x \ln x}$$

   In order to compute $\lim_{x\to 0^+} \sin x \ln x$, we first write:

   $$\sin x \ln x = \frac{\ln x}{\frac{1}{\sin x}}$$

   and note that $\lim_{x\to 0^+} \ln x = -\infty$, $\lim_{x\to 0^+} \frac{1}{\sin x} = +\infty$. Therefore, we may apply L' Hôpital's rule and consider the limit:

   $$\lim_{x\to 0^+} \frac{(\ln x)'}{\left(\frac{1}{\sin x}\right)'} = \lim_{x\to 0^+} \frac{\frac{1}{x}}{\frac{-\cos x}{\sin^2 x}} = \lim_{x\to 0^+} \frac{\sin^2 x}{x\cos x}$$

   This last limit is an indeterminacy of the form $0/0$. Therefore we may apply L' Hôpital's rule and get the limit:

   $$\lim_{x\to 0^+} \frac{(\sin^2 x)'}{(x\cos x)'} = \lim_{x\to 0^+} \frac{2\sin x\cos x}{\cos x - x\sin x}$$

   This last limit can be computed with direct evaluation and it equals 0. Therefore:

$$\lim_{x\to 0^+} \sin x \ln x = 0$$

and thus:

$$\lim_{x\to 0^+} x^{\sin x} = e^0 = 1$$

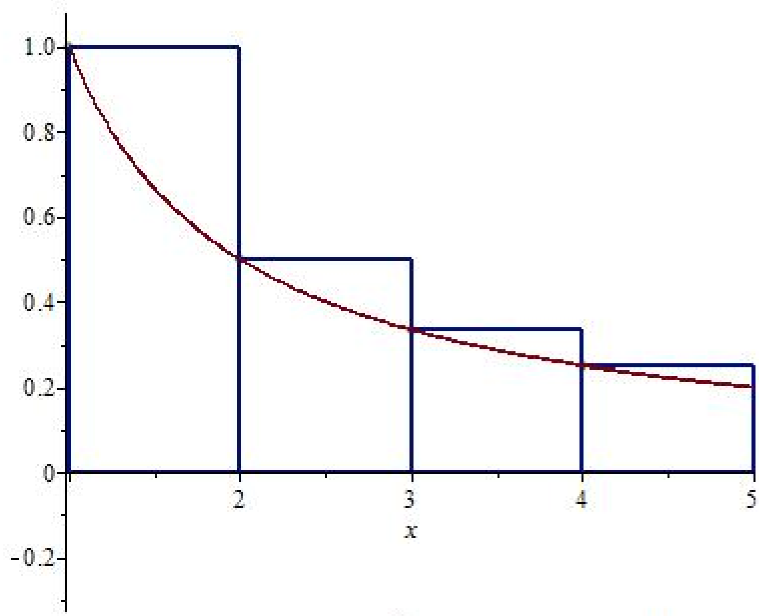
2. Let $f(x) = \dfrac{1}{x}$ on $[1, 5]$; $n = 4$.

   (a) Illustrate the left and right Riemann sums for f on the given interval and for the given value of n. Determine which Riemann sum underestimates and which sum overestimates the area under the curve.

   (b) Calculate the left and right Riemann sums.

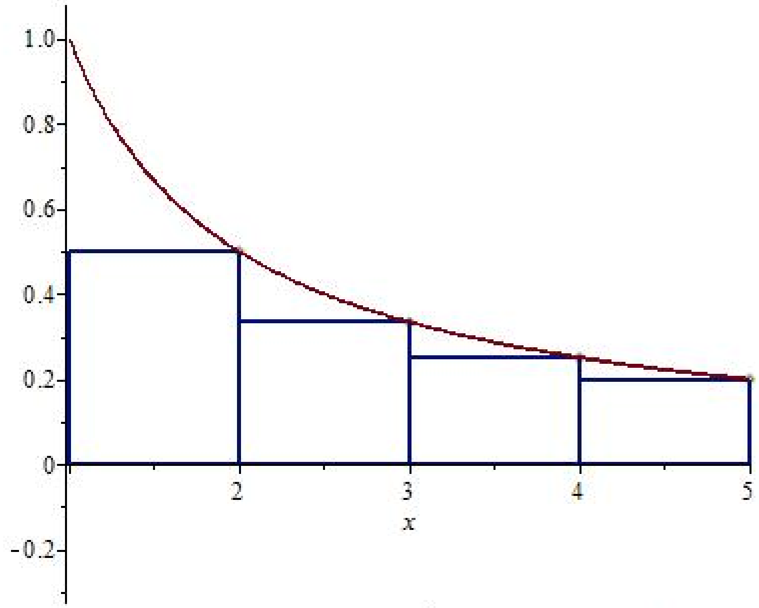
   SOLUTION:

   (a) Left-hand illustration:



A left Riemann sum approximation of $\int_1^5 f(x)\, dx$, where $f(x) = \dfrac{1}{x}$ and the partition is uniform. The approximate value of the integral is 2.083333333. Number of subintervals used: 4.

Right-hand illustration:



A right Riemann sum approximation of $\int_1^5 f(x)\,dx$, where $f(x) = \frac{1}{x}$ and the partition is uniform. The approximate value of the integral is 1.283333333. Number of subintervals used: 4.

The left Riemann sum is an overestimate and the right Riemann sum is a underestimate.

(b)

$$\begin{aligned}
\text{Left Riemann sum} &= \sum_{k=1}^{4} f(\overline{x_k})\Delta x \\
&= f(1)\cdot 1 + f(2)\cdot 1 + f(3)\cdot 1 + f(4)\cdot 1 \\
&= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} \\
&= \frac{25}{12}
\end{aligned}$$

$$\begin{aligned}
\text{Right Riemann sum} &= \sum_{k=1}^{4} f(\overline{x_k})\Delta x \\
&= f(2)\cdot 1 + f(3)\cdot 1 + f(4)\cdot 1 + f(5)\cdot 1 \\
&= \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} \\
&= \frac{77}{60}
\end{aligned}$$

3. Each of the sums below is written in sigma notation. Expand it to a sum in which each term is written separately. (For example, $\sum_{n=1}^{4} 2n$ would become $2+4+6+8$.)

   (a) $\displaystyle\sum_{n=0}^{5} \frac{1}{(-2)^n}$

   (b) $\displaystyle\sum_{n=-3}^{3} \frac{n-1}{(n^2+1)}$

   (c) $\displaystyle\sum_{n=1}^{6} (-n)^{n-2} \cos\left(\frac{n\pi}{2}\right)$

   SOLUTION:

   (a) $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16} - \frac{1}{32}$

   (b) $-\frac{4}{10} - \frac{3}{5} - \frac{2}{2} - \frac{1}{1} + \frac{0}{2} + \frac{1}{5} + \frac{2}{10}$

   (c) $0 - 1 + 0 + 16 + 0 - 1296$

4. Identify a pattern in the terms of each sum written below and use the pattern to convert the sum to sigma notation. (For example, $2+4+6+8$ could become $\sum_{n=1}^{4} 2n$ or $\sum_{n=0}^{3} 2(n+1)$.)

   (a) $3+6+11+18+27+38+51+66+83+102+123+146+171+198+227+258+291+326+363+402$

   (b) $\frac{1}{9} - \frac{1}{3} + 1 - 3 + 9 - 27 + 81 - 243$

   (c) $\frac{\ln(11)}{23} + \frac{\ln(13)}{28} + \frac{\ln(15)}{33} + \frac{\ln(17)}{38} + \frac{\ln(19)}{43} + \frac{\ln(21)}{48}$

   SOLUTION:

   (a) We notice the presence of 102 and 402, which are 2 greater than perfect squares ($102 = 10^2 + 2$, $402 = 20^2 + 2$). We then check that all of the terms have this form, starting with $3 = 1^2 + 2$ and ending with $402 = 20^2 + 2$. Thus the general term is $n^2 + 2$ and n ranges from 1 to 20, and the sum can be expressed as

$$\sum_{n=1}^{20} (n^2 + 2).$$

(b) The terms are powers of 3, starting with $1/9 = 1/3^2 = 3^{-2}$ and ending with $243 = 3^5$. Also, the signs alternate $+ - +-$, which is equivalent to taking powers of (-3). The sum can therefore be expressed as

$$\sum_{n=-2}^{5} (-3)^n.$$

(c) We notice that each numerator is the natural logarithm of an odd number. Odd numbers can be written as $2n+1$, where n is an integer. The sum starts with numerator $\ln(11) = \ln(2 \times 5 + 1)$ and ends with $\ln(23) = \ln(2 \times 11 + 1)$, so the numerators are $\ln(2n + 1)$ with n ranging from 5 to 10.

Also notice that the denominator increases by 5 with each successive term, so we expect the denominator could be expressed in terms of $5n$. The first denominator is 23, and this is supposed to be the term with $n = 5$, so $23 = 5 \times 5 - 2$ suggests the denominator has the form $5n - 2$. We check that this works for all of the other terms.

Thus the sum can be expressed as

$$\sum_{n=5}^{10} \frac{\ln(2n+1)}{5n-2}.$$

Note: For a mass m moving along the x axis, the equations of motion give the acceleration $a(t)$, velocity $v(t)$ and the displacement $s(t)$ as a function of time t.

- acceleration is defined as $a(t) = \frac{dv}{dt}$
- velocity is defined as $v(t) = \frac{ds}{dt}$
- a, v and s can be positive (in the positive x direction) or negative (in the negative x direction).

5. A mass m is moving along the positive x direction with constant acceleration $a(t) = -9.8$.

Find the equations of motion:

(a) $v(t)$ if $v(0) = 3$; and

(b) $s(t)$ if $v(0) = 3$ and $s(0) = 4$.

SOLUTION: (a)

$$v(t) = \int -9.8 \; dt = -9.8t + C$$

To find C, we use the condition $v(0) = 3$:

$$\begin{aligned} v(0) &= -9.8(0) + C \\ 3 &= C. \end{aligned}$$

The velocity equation is then $v(t) = -9.8t + 3$.

(b)

$$s(t) = \int (-9.8t + 3)\ dt = -4.9t^2 + 3t + C$$

To find C, we use the condition $s(0) = 4$:

$$\begin{aligned} s(0) &= -4.9(0)^2 + 3(0) + C \\ 4 &= C. \end{aligned}$$

The displacement equation is then $s(t) = -4.9t^2 + 3t + 4$.

6. Mass m is moving along the x axis with acceleration

$$a(t) = 2\cos(3t).$$

Find the equations of motion:

(a) $v(t)$ if $v(0) = 4$; and

(b) $s(t)$ if $v(0) = 4$ and $s(0) = 5$.

Solution: (a)

$$v(t) = \int 2\cos(3t)\ dt = \frac{2}{3}\sin(3t) + C$$

To find C, we use the condition $v(0) = 4$:

$$\begin{aligned} v(0) &= \frac{2}{3}\sin(3(0)) + C \\ 4 &= 0 + C. \end{aligned}$$

The velocity equation is then $v(t) = \frac{2}{3}\sin(3t) + 4$.

(b)

$$s(t) = \int \left(\frac{2}{3}\sin(3t) + 4\right)\ dt = -\frac{2}{9}\cos(3t) + 4t + C$$

To find C, we use the condition $s(0) = 5$:

$$\begin{aligned} s(0) &= -\frac{2}{9}\cos(3(0)) + 4(0) + C \\ 5 &= -\frac{2}{9} + C \\ C &= \frac{47}{9}. \end{aligned}$$

The displacement equation is then $s(t) = -\dfrac{2}{9}\cos(3t) + 4t + \dfrac{47}{9}$.

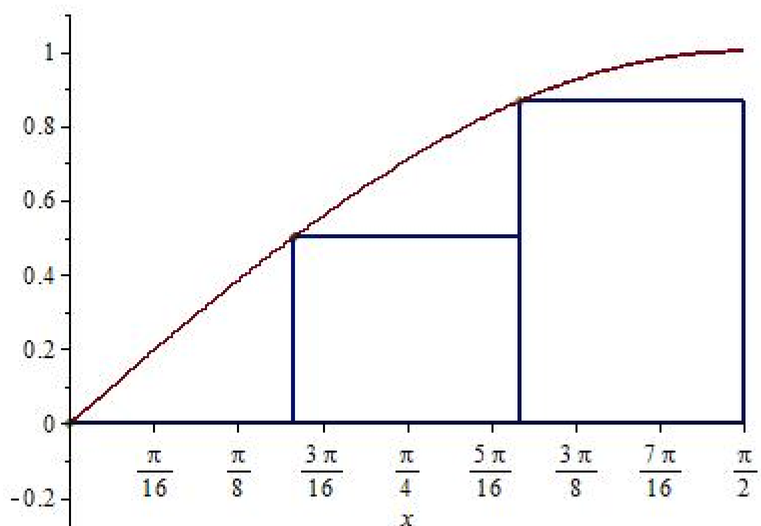
7. Let R be the region bounded by the graph of $f(x) = \sin x$ and the x-axis between $x = 0$ and $x = \pi/2$.

(a) Approximate the area of R using a left Riemann sum with $n = 3$ subintervals. Evaluate all trigonometric functions. Illustrate the sum with the appropriate rectangles.

(b) Approximate the area of R using a right Riemann sum with $n = 3$ subintervals. Evaluate all trigonometric functions. Illustrate the sum with the appropriate rectangles.

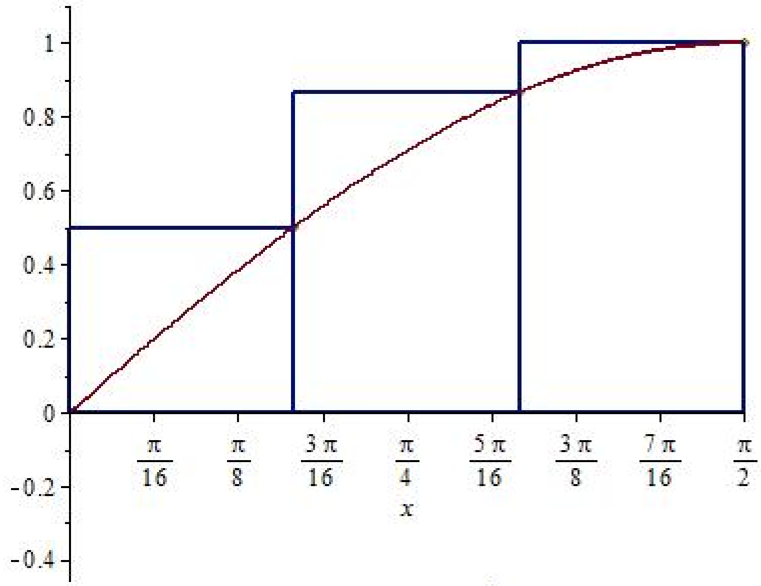
(c) How do the area approximations in parts (a) and (b) compare to the actual area under the curve?

SOLUTION: (a) $L_3 = \cos(0)\cdot\frac{\pi}{6} + \cos(\pi/6)\frac{\pi}{6} + \cos(\pi/3)\frac{\pi}{6} = \frac{(3+\sqrt{3})\pi}{12}$.



A left Riemann sum approximation of $\int_0^{\frac{1}{2}\pi} f(x)\,dx$, where $f(x) = \sin(x)$ and the partition is uniform. The approximate value of the integral is 0.7152492291. Number of subintervals used: 3.

(b) $R_3 = \cos(\pi/6)\cdot\frac{\pi}{6} + \cos(\pi/3)\frac{\pi}{6} + \cos(\pi/2)\frac{\pi}{6} = \frac{(1+\sqrt{3})\pi}{12}$.



A right Riemann sum approximation of $\int_0^{\frac{1}{2}\pi} f(x)\,dx$, where $f(x) = \sin(x)$ and the partition is uniform. The approximate value of the integral is 1.238848005. Number of subintervals used: 3.

(c) The left Riemann sum in part (a) underestimates the actual area of R, while the right Riemann sum in part (b) overestimates the area of R.

8. Calculate the following indefinite integrals

(a) $\displaystyle\int \csc^2 x \; dx$

(b) $\displaystyle\int 2^t \; dt$

(c) $\displaystyle\int \left(\sqrt{x} - 3\sqrt[5]{x}\right) \; dx$

(d) $\displaystyle\int \left(e^{2t} + \frac{1}{2t}\right) \; dt$

SOLUTION: (a) Since $\dfrac{d}{dx}(\cot x) = -\csc^2 x$, we see that

$$\int \csc^2 x \; dx = -\cot x + C.$$

(b) Since $\frac{d}{dt}(2^t) = 2^t \ln(2)$, we see that

$$\int 2^t \ dt = \frac{1}{\ln(2)} 2^t + C.$$

(c) $\int \left(\sqrt{x} - 3\sqrt[5]{x}\right) \ dx = \int \left(x^{1/2} - 3x^{1/5}\right) \ dx = \frac{2}{3}x^{3/2} - \frac{15}{6}x^{6/5} + C$

(d) $\int \left(e^{2t} + \frac{1}{2t}\right) \ dt = \int \left(e^{2t} + \frac{1}{2} \cdot \frac{1}{t}\right) \ dt = \frac{1}{2}e^{2t} + \frac{1}{2}\ln(t) + C$

Below are the problems that were graded and the scoring system that was used for each problem.

2. [10 points] Let $f(x) = \dfrac{1}{x}$ on $[1, 5]$; $n = 4$.
(a) [4 points] Illustrate the left and right Riemann sums for f on the given interval and for the given value of n. Determine which Riemann sum underestimates and which sum overestimates the area under the curve.
2 points – If the illustration for the left Riemann sum is correct
2 points – If the illustration for the right Riemann sum is correct

(b) [6 points] Calculate the left and right Riemann sums.
2 points – If the student correctly set up the left Riemann sum
3 points – If the student correctly set up the left Riemann sum AND computed it correctly (they need to have add the terms together)
Grade the right Riemann sum the same way.

6. Mass m is moving along the x axis with acceleration

$$a(t) = 2\cos(3t).$$

(a) Find $v(t)$ if $v(0) = 4$.
3 points – If the student correctly finds that $v(t) = \dfrac{2}{3}\sin(3t) + C$
2 points – If the student uses $v(0) = 4$ to find the value for C.
[Note: Even if the student does not find $v(t)$ correctly, s/he can still earn 2 points for correctly finding C using their $v(t)$ and the condition $v(0) = 4$.]

8. (c) [5 points] $\displaystyle\int \left(\sqrt{x} - 3\sqrt[5]{x}\right)\, dx$

1 point – If the student correctly rewrote the integral as $\displaystyle\int \left(x^{1/2} - 3x^{1/5}\right)\, dx$
3 points – If the student correctly rewrote the integral AND integrated only one of the terms correctly using the power rule
5 points – If the student correctly rewrote the integral AND integrated both of the terms correctly using the power rule
-1 point – If the student forgot $+C$

8. (d) [5 points] $\displaystyle\int \left(e^{2t} + \frac{1}{2t}\right)\, dt$

2 points – If the student integrated only one of the terms correctly

5 points – If the student integrated both of the terms correctly

-1 point – If the student forgot $+C$