Math 180 Written Homework
Assignment #8
Due **Tuesday, November 11th** at the beginning of your discussion class.

Directions. You are welcome to work on the following problems with other MATH 180 students, but your solutions must be hand-written, by your own hand, representing your understanding of the material. Word-by-word copying from another student or any other source is unacceptable. Any work without the proper justification will receive no credit. The list of problem solutions is to be submitted to your TA at the beginning of the discussion class listed above. No late homework will be accepted.

1. Let $f(x) = \ln(2/x)$.

   (a) Compute the best linear approximation of f at $x = 2$.

   (b) Use the best linear approximation from part (a) to estimate the value of $\ln(1.9)$.

   (c) Determine whether your estimate of $\ln(1.9)$ in part (b) is an overestimate or an underestimate of the actual value of $\ln(1.9)$. Use calculus to justify your answer, not a calculator.

   Solution:

   (a) The best linear approximation of f at $x = 2$ is given by the formula

   $$L(x) = f(2) + f'(2)(x - 2).$$

   The derivative of f is $f'(x) = -\dfrac{1}{x}$. The values of f and f' at $x = 2$ are

   $$f(2) = \ln(2/2) = \ln(1) = 0, \qquad f'(2) = -\frac{1}{2}.$$

   Thus, the best linear approximation is

   $$L(x) = 0 - \frac{1}{2} \cdot (x - 2) = -\frac{1}{2}(x - 2).$$

   (b) Notice that $f(x) = \ln(1.9)$ when $x = \dfrac{2}{1.9}$. So the estimated value of $\ln(1.9)$ using the best linear approximation from part (a) is

   $$\ln(1.9) \approx L\left(\frac{2}{1.9}\right) = -\frac{1}{2}\left(\frac{2}{1.9} - 2\right) = \frac{9}{19}.$$

   (c) Since $f''(x) = \dfrac{1}{x^2} > 0$ on $(0, \infty)$, the graph is concave up on the interval so the tangent line at $x = 2$ lies below the graph of f at $x = 1.9$. Therefore, the estimate in part (b) is an underestimate to the actual value of $\ln(1.9)$.

2. Sketch the graph of the function $f(x) = \frac{1}{3}x - (x-1)^{1/3}$. Find and label all critical points, local maxima, local minima, inflections, and asymptotes.

Note: Your graph must include the y-intercept, but it is not necessary to find and label the x-intercepts. There are no horizontal asymptotes.

SOLUTION: We follow the steps in the textbook.

1. Domain: The domain of this function is all real numbers x. (The fractional power $(x-1)^{1/3}$ has odd denominator, so there is no restriction on its domain.)

2. Symmetry: The function is neither odd nor even, so the graph will have no special symmetry.

3. Derivatives: We calculate

$$f'(x) = \frac{1}{3} - \frac{1}{3(x-1)^{2/3}}$$

and

$$f''(x) = \frac{2}{9(x-1)^{5/3}}.$$

4. Critical points and possible inflections: The first derivative is not defined at $x = 1$, which is in the domain of f, so this is a critical point. To find other critical points we must solve

$$f'(x) = \frac{1}{3} - \frac{1}{3(x-1)^{2/3}} = 0$$

or equivalently

$$\frac{1}{3(x-1)^{2/3}} = \frac{1}{3}.$$

Multiplying by 3 and taking the reciprocal of both sides, we have

$$(x-1)^{2/3} = 1$$

and thus

$$x - 1 = \pm 1.$$

Thus the two other critical points are $x = 0$ and $x = 2$.

The second derivative is not defined at $x = 1$, which is in the domain of f, so this is a possible inflection point. To find other possible inflections we must solve

$$f''(x) = \frac{2}{9(x-1)^{5/3}} = 0$$

but this has no solutions. Therefore $x = 1$ is the only possible inflection point.

5. Intervals of increase/decrease and intervals of concavity: The critical points $0, 1, 2$ determine four intervals on which we need to know the sign of the derivative: $(-\infty, 0)$, $(0, 1)$, $(1, 2)$, and $(2, \infty)$. We can find these signs by testing values in each interval. For this function, it is helpful to take some care in selecting test values where $(x-1)^{2/3}$ is simple to evaluate. This will be the case when $(x-1)$ is a perfect cube, or the reciprocal of one. Choosing test values like this, we find

- $f'(-7) = \frac{1}{3} - \frac{1}{3(-8)^{2/3}} = \frac{1}{3} - \frac{1}{12} = \frac{1}{4} > 0$
  Thus $f' > 0$ on the interval $(-\infty, 0)$.
- $f'(7/8) = \frac{1}{3} - \frac{1}{3(-1/8)^{2/3}} = \frac{1}{3} - \frac{4}{3} = -1$
  Thus $f' < 0$ on the interval $(0, 1)$.
- $f'(9/8) = \frac{1}{3} - \frac{1}{3(1/8)^{2/3}} = \frac{1}{3} - \frac{4}{3} = -1$
  Thus $f' < 0$ on the interval $(1, 2)$.
- $f'(9) = \frac{1}{3} - \frac{1}{3(8)^{2/3}} = \frac{1}{3} - \frac{1}{12} = \frac{1}{4} > 0$
  Thus $f' > 0$ on the interval $(2, \infty)$.

Therefore, the function is increasing on $(-\infty, 0)$ and $(2, \infty)$ and decreasing on $(0, 1)$ and $(1, 2)$.

Turning to the second derivative, the only possible inflection point at $x = 1$ gives intervals $(-\infty, 1)$ and $(1, \infty)$ we must test. Since $f''(x)$ has the same sign as $(x-1)$, we find that f is concave up on $(1, \infty)$ and concave down on $(-\infty, 1)$.

6. Identify extreme values and inflections: By the first derivative test, the increasing/decreasing intervals determined above show that $x = 0$ is a local maximum, $x = 2$ is a local minimum, and $x = 1$ is not a local extreme point. Calculating y values we have

- $(0, f(0)) = (0, 1)$ is a local maximum
- $(2, f(2)) = (2, -\frac{1}{3})$ is a local minimum

We also see that the concavity changes at $x = 1$, so

- $(1, f(1)) = (1, \frac{1}{3})$ is an inflection point

and since $\lim_{x \to 1} f'(x) = -\infty$, at this inflection point the graph has a vertical tangent.

7. Asymptotes and end behavior: Since f is defined for all x, it has no vertical asymptotes. The problem tells us that there are no horizontal asymptotes, which we could also determine by showing

$$\lim_{x \to \infty} f(x) = \infty$$

(because $x/3$ grows faster than $(x-1)^{1/3}$) and

$$\lim_{x \to -\infty} f(x) = -\infty$$

(for essentially the same reason).

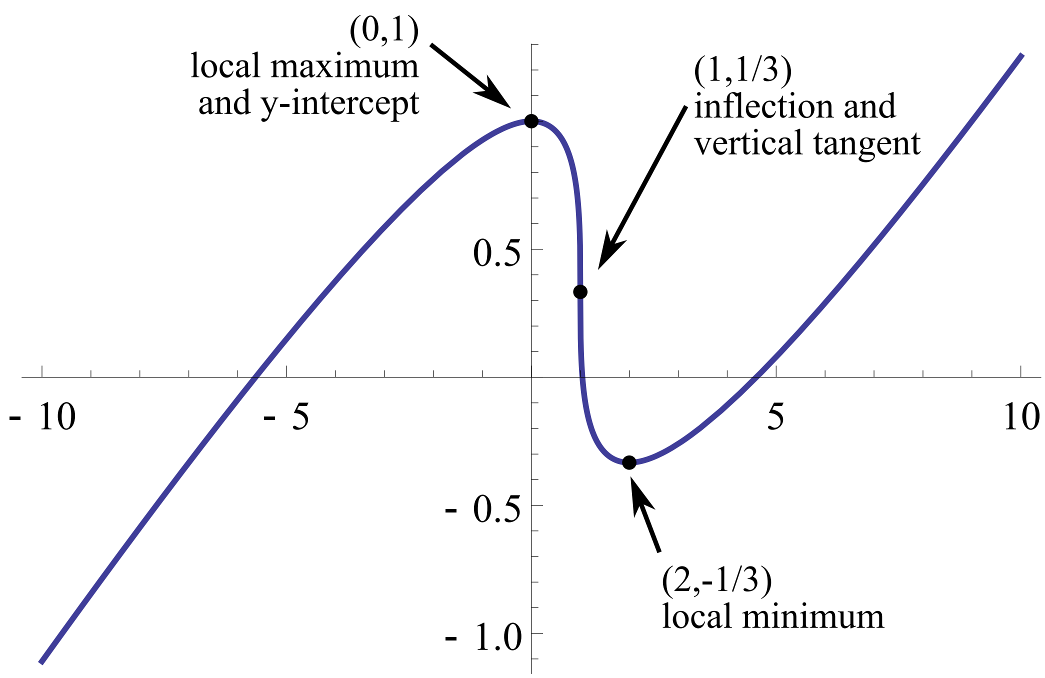
8. Intercepts: The y-intercept is $(0, f(0)) = (0, 1)$ which is also a local maximum. Using a calculator we can find approximate x-intercepts $(-5.63, 0)$, $(1.04, 0)$, $(4.60, 0)$, but the problem does not require this.

9. Choose a window and make a graph: There are many possible windows in which to sketch the graph. The x range needs to include $0, 1, 2$ and the y range should include all of the values we have considered, $-\frac{1}{3}, 0, \frac{1}{3}$. We choose a larger window $[-10, 10]$ on the x axis and $[-2, 2]$ on the y axis, which will show all of the intercepts.

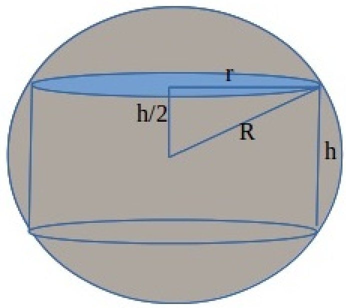
We need to sketch a graph that is

- Increasing and concave down on the interval $(-\infty, 0)$
- Has a local max at the point $(0, 1)$
- Is decreasing and concave down on the interval $(0, 1)$
- Has a vertical tangent and inflection at the point $(1, \frac{1}{3})$
- Is decreasing and concave up on the interval $(1, 2)$
- Has a local minimum at the point $(2, -\frac{1}{3})$
- Is increasing and concave up on the interval $(2, \infty)$.

Incorporating all of these features, we find a graph as below:



3. Find the volume of the largest right circular cylinder of radius r and height h that can be inscribed in a sphere of radius R.



Solution: The volume of a right cylinder is the area of one end times the height, $V = A \cdot h$.
From the right triangle in the picture $R^2 = r^2 + \frac{h^2}{4}$ or $r^2 = R^2 - \frac{h^2}{4}$

$$\begin{aligned} V(h) &= \pi\left(R^2 - \frac{h^2}{4}\right)h \\ &= \pi\left(R^2 h - \frac{h^3}{4}\right), 0 \leq h \leq 2R \\ V'(h) &= \pi\left(R^2 - \frac{3h^2}{4}\right) = 0 \\ h^2 &= \frac{4R^2}{3}, \\ h &= \frac{2R}{\sqrt{3}}. \end{aligned}$$

Thus $h = \frac{2R}{\sqrt{3}}$ is the only critical value.
Use the second derivative test to verify that the critical point gives a local maximum.
$V''(h) = -\dfrac{6h\pi}{4}$ and $V''\left(\dfrac{2R}{\sqrt{3}}\right) < 0$ so the critical point gives a maximum volume.
At the end points $h = 0$ and $h = 2R$ the volume collapses to zero so the critical point gives the absolute maximum volume. The maximum volume is $V\left(\dfrac{2R}{\sqrt{3}}\right) = \dfrac{4\pi R^3}{3\sqrt{3}}$.

4. The radius of a cylinder is decreasing at a rate of 4 ft/min, while the height is increasing at a rate of 2 ft/min. Find the rate of change in the volume when the radius is two feet and the height is six feet.

SOLUTION: We know that $\dfrac{dr}{dt} = -4$ ft/min and $\dfrac{dh}{dt} = 2$ ft/min.

The unknown is $\dfrac{dV}{dt}$, which we want to determine at the moment when $r = 2$ ft and $h = 6$ ft.

From the formula for the volume of a cylinder, $V = \pi r^2 h$, we obtain

$$\begin{aligned} \frac{dV}{dt} &= \pi r^2 \frac{dh}{dt} + 2\pi r h \frac{dr}{dt} \\ &= \pi(2)^2(2) + 2\pi(2)(6)(-4), \text{ when } r = 2 \text{ and } h = 6, \\ &= -88\pi \text{ ft}^3/\text{min}. \end{aligned}$$

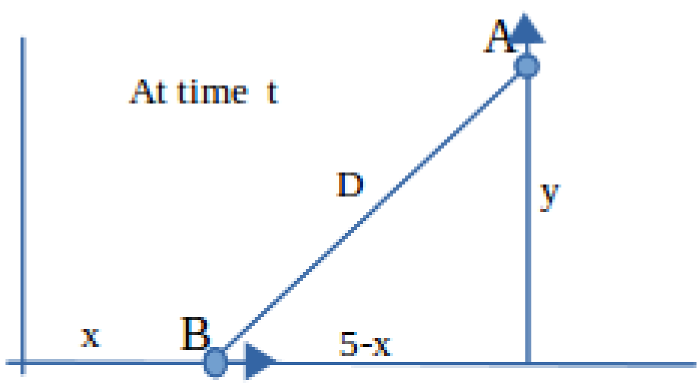
The minus indicates that the volume is decreasing.

5. Ship A is steaming north at 10 miles per hour. Ship B, which is 5 miles west of ship A, is steaming east at 15 miles per hour.

(a) At what rate is the distance between them changing?

(b) At what rate will it be changing one hour from now?

SOLUTION: Consider the following picture.



We can see that $D^2 = (5-x)^2 + y^2$, and we're given that $\dfrac{dx}{dt} = 15$ mph and $\dfrac{dy}{dt} = 10$ mph.

(a) At $t = 0$, $D = 5$ mi, $x = 0$ and $y = 0$. We're looking for $\dfrac{dD}{dt}$. Therefore

$$\begin{aligned} 2D \cdot \frac{dD}{dt} &= 2(5-x)\cdot\left(-\frac{dx}{dt}\right) + 2y\cdot\frac{dy}{dt} \\ 2(5)\frac{dD}{dt} &= 2(5)(-15) + 2(0)(10) \\ \frac{dD}{dt} &= -15 \text{ mph} \end{aligned}$$

The negative sign implies the distance between the ships is decreasing at $t = 0$.

(b) At $t = 1$, $x = 15$, $y = 10$, so $D = 10\sqrt{2}$ and again we're looking for $\dfrac{dD}{dt}$. Therefore

$$\begin{aligned} 2D \cdot \frac{dD}{dt} &= 2(5-x)\cdot\left(-\frac{dx}{dt}\right) + 2y\cdot\frac{dy}{dt} \\ 2(10\sqrt{2})\frac{dD}{dt} &= 2(-10)(-15) + 2(10)(10) \\ \frac{dD}{dt} &= \frac{25\sqrt{2}}{2} \approx 17.68 \text{ mph} \end{aligned}$$

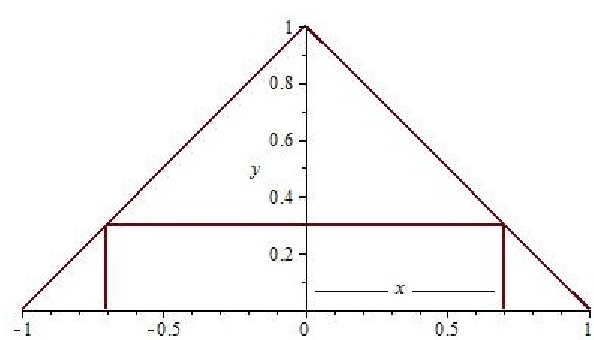
Note, this answer makes sense because after one hour, ship B is east of where ship A started and the ships are moving apart.

6. Consider the function $g(x) = -|x| + 1$ for $-1 \le x \le 1$.

   (a) Draw g on the domain given.

   (b) A rectangle is to be formed with one side on the x-axis, and the vertices of the opposite side lying on g. Find the maximum area of such a rectangle.

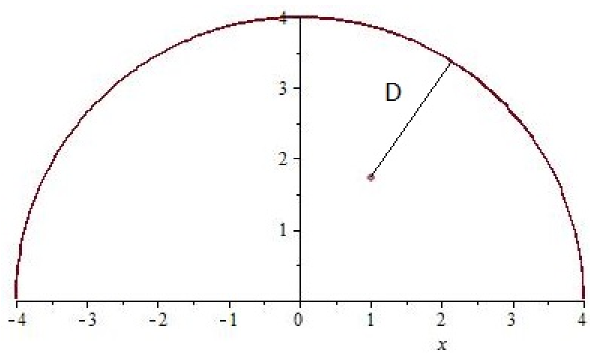
SOLUTION: (a)



The area of the rectangle is $A(x) = 2x \cdot g(x) = 2x(-x+1) = -2x^2 + 2x$ for $0 < x < 1$. Then $A'(x) = -4x + 2 = 0$, so $x = \frac{1}{2}$. Since $A''(x) = -4 < 0$, by the second derivative test, this critical point is a local (and hence absolute) maximum. The maximum area of such a rectangle is then $A\left(\frac{1}{2}\right) = \frac{1}{2}$.

7. How close does the semicircle $y = \sqrt{16 - x^2}$ come to the point $\left(1, \sqrt{3}\right)$?

SOLUTION: Instead of minimizing the distance between the point and the semicircle, we'll minimize the *square* of the distance between them. The reason we are able to do this is because if d_1 and d_2 are two positive numbers, then $d_1^2 > (d_2)^2$ exactly when $d_1 > d_2$, so a minimum of the square of the distance is also a minimum of the distance itself. The reason for minimizing the square of the distance is to greatly simplify our derivative. [See Example 4 on page 274 for another example of minimizing the square of the distance.]



The distance D between a point (x, y) and $\left(1, \sqrt{3}\right)$ satisfies

$$\begin{aligned} D^2 &= (x-1)^2 + (y - \sqrt{3})^2 \\ D^2 &= (x-1)^2 + \left(\sqrt{16 - x^2} - \sqrt{3}\right)^2 \end{aligned}$$

We'll minimize D^2 for $-4 \le x \le 4$. Multiplying out, we have

$$\begin{aligned} D^2 &= -2x + 20 - 2\sqrt{3}\sqrt{16 - x^2} \\ (D^2)' &= -2 + \frac{2\sqrt{3}x}{\sqrt{16 - x^2}} = 0 \\ 4\sqrt{16 - x^2} &= 2\sqrt{3}x \\ 4(16 - x^2) &= 12x^2 \\ x &= 2. \end{aligned}$$

If we plug in $x = 2$ and the endpoints $x = \pm 4$ in to D^2, we find that

$$\begin{aligned} x &= 2 & D^2 &= 4 \\ x &= -4 & D^2 &= 28 \\ x &= -4 & D^2 &= 12 \end{aligned}$$

So the closest the semicircle comes to the point $(1, \sqrt{3})$ is 4 units.

8. A marathon runner ran the Chicago marathon (which is 26.2 miles long) in 2 hours and 12 minutes. Explain (using calculus) how you know the runner had to be running at exactly 11 miles per hour at least twice during the race.

SOLUTION: The runner ran the marathon at an average pace of 26.2 miles divides by 2 hours and 12 minutes, or 11.91 mph. Since the runner began the race from rest (i.e. 0 mph), by the Intermediate Value Theorem the runner must have hit 11 mph at some point when reaching the average pace of 11.91 mph. If we also assume that the runner stopped running at the end of the race, then, again using the Intermediate Value Theorem, when coming back down from the average pace of 11.91 mph to 0 mph, the runner must have also been going 11 mph at some point.

9. (a) Show that the best linear approximation of $f(x) = (1+x)^k$ at $x = 0$ is $L(x) = 1 + kx$.

   (b) Use your answer in part (a) to find the best linear approximation for the functions $f(x)$ at $x = 0$.

   - $f(x) = (1-x)^6$
   - $f(x) = (4+3x)^{1/3}$

   (c) Using your answer in part (a), estimate $\sqrt[4]{1.009}$. Write your estimate in the form $\frac{a}{b}$ where a and b are integers.

SOLUTION: (a) Since $f'(x) = k(1+x)^{k-1}$, we find that

$$L(x) = f(0) + f'(0)(x-0) = 1 + kx.$$

(b) If $f(x) = (1-x)^6 = (1+(-x))^6$, then using part (a), we find that the best linear approximation for f is

$$L(x) = 1 + 6(-x) = 1 - 6x.$$

If $f(x) = (4+3x)^{1/3}$, then we can rewrite f as $f(x) = 4^{1/3} \cdot \left(1 + \left(\frac{3x}{4}\right)\right)^{1/3}$. Now replace k with $1/3$, x with $3x/4$ and multiply by $4^{1/3}$ to get the best linear approximation for f of

$$L(x) = 4^{1/3}\left(1 + \frac{1}{3} \cdot \frac{3x}{4}\right) = 4^{1/3}\left(1 + \frac{x}{4}\right).$$

(c) Here $f(x) = \sqrt[4]{1+x} = (1+x)^{1/4}$ which can be approximated using $L(x) = 1 + \frac{1}{4}x$. Then

$$\sqrt[4]{1.009} = f(0.009) \approx L(0.009) = 1 + \frac{1}{4}(0.009) = \frac{4009}{4000}.$$

10. Consider the function $h(x) = \dfrac{|x-1|}{x+2}$.

(a) Write h as a piecewise function.

(b) State the domain of h.

(c) Find and classify the critical points of h.

(d) Find the intervals of increase and decrease for h.

(e) Find any vertical asymptotes of h.

(f) Find all horizontal asymptotes of h.

(g) Using your information from the parts above, sketch a graph of h.

SOLUTION:

(a) Write h as a piecewise function.

First of all,

$$|x-1| = \begin{cases} x-1 & \text{if } x-1 \geq 0 \\ -(x-1) & \text{if } x-1 < 0 \end{cases} = \begin{cases} x-1 & \text{if } x \geq 1 \\ 1-x & \text{if } x < 1 \end{cases}$$

Substituting this into the formula for h we have

$$h(x) = \begin{cases} \frac{x-1}{x+2} & \text{if } x \geq 1 \\ \frac{1-x}{x+2} & \text{if } x < 1 \end{cases}$$

(b) State the domain of h.

The numerator and denominator are defined for all x, so the domain of h is all values of x except those at which the denominator is zero. The only point where $x+2=0$ is $x=-2$, so the domain is all x except -2 (equivalently, it is $(-\infty,-2)$ and $(-2,\infty)$).

(c) Find and classify the critical points of h.

We calculate the derivative. Note that

$$\frac{d}{dx}\frac{x-1}{x+2}=\frac{(x+2)(1)-(x-1)(1)}{(x+2)^2}=\frac{3}{(x+2)^2}$$

so we have

$$h'(x)=\begin{cases}\frac{3}{(x+2)^2} & \text{if } x>1\\ ?? & \text{if } x=1\\ \frac{-3}{(x+2)^2} & \text{if } x<1\end{cases}$$

We have left the case $x=1$ undetermined because it is where the two parts of the piecewise definition come together. It might be a corner or the derivative might exist there, depending on whether the functions used to either side of $x=1$ have the same derivative at $x=1$ or not.

However, substituting $x=1$ into $\frac{3}{(x+2)^2}$ and $\frac{-3}{(x+2)^2}$ gives $1/3$ and $-1/3$, respectively, so $x=1$ is a corner point and

$$h'(x)=\begin{cases}\frac{3}{(x+2)^2} & \text{if } x>1\\ \text{does not exist} & \text{if } x=1\\ \frac{-3}{(x+2)^2} & \text{if } x<1\end{cases}.$$

Neither part of the piecewise definition of $h'(x)$ has any zeros, since each is a rational function with constant numerator. However, at $x=1$ the derivative does not exist, so this is the only critical point. (The point $x=-2$ is not a critical point because it is not in the domain of h.)

It will be determined in the next part of the problem that h changes from decreasing to increasing as x increases through 1, so it follows that $(1,h(1))=(1,0)$ is a local minimum.

(d) Find the intervals of increase and decrease for h.

The points where h could change between increasing and decreasing are $x=-2$ (since it is not in the domain) and $x=1$ (since it is a critical point). Thus we need to test a value in each interval $(-\infty,-2)$, $(-2,1)$, and $(1,\infty)$. Using the formula above we find

- $h'(-3)=-3$, so h is decreasing on $(-\infty,-2)$

- $h'(0) = -3/4$, so h is decreasing on $(-2, 1)$
- $h'(2) = 3/16$, so h is increasing on $(1, \infty)$

(e) Find any vertical asymptotes of h

Since $x = -2$ is a zero of the denominator of h but not of the numerator, the line $x = -2$ is a vertical asymptote. Since $h(x) = \frac{1-x}{x+2}$ for x near -2, the function is negative to the left of -2 and positive to the right, and we have

- $\lim_{x\to-2^-} h(x) = -\infty$
- $\lim_{x\to-2^+} h(x) = \infty$

(f) Find all horizontal asymptotes of h

For large positive x, we have $h(x) = \frac{x-1}{x+2}$ which is a rational function with numerator and denominator of equal degree. The limit as $x \to \infty$ is therefore the ratio of the coefficients of the top degree terms, which in this case is:

$$\lim_{x\to\infty} h(x) = \frac{1}{1} = 1.$$

For large negative x, we have $h(x) = \frac{1-x}{x+2}$, and again the limit is the ratio of coefficients:

$$\lim_{x\to\infty} h(x) = \frac{-1}{1} = -1.$$

Therefore both of the lines $y = 1$ and $y = -1$ are horizontal asymptotes.

(g) Using our information from the parts above, sketch a graph of h.

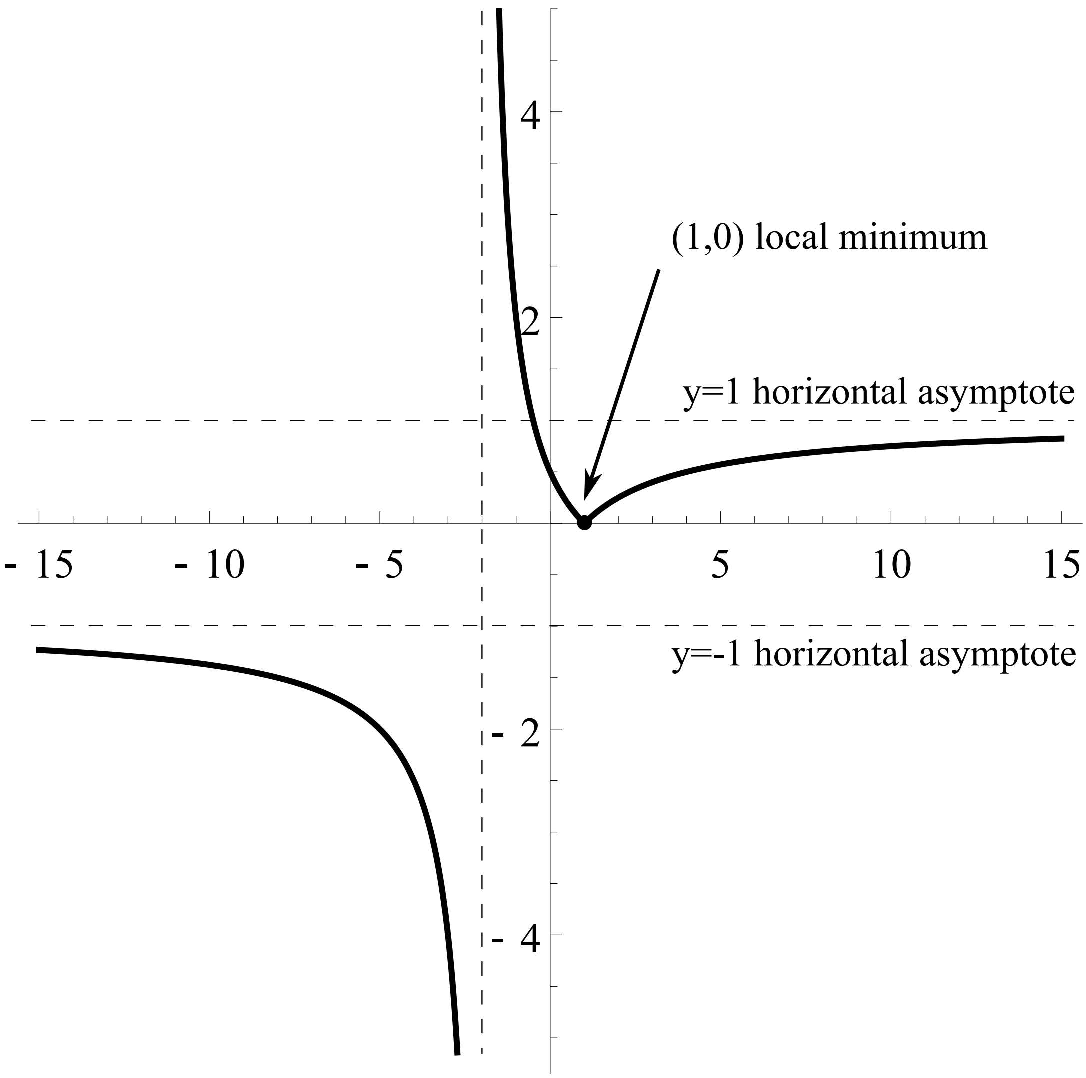
There are many possible windows in which to sketch the graph. A range of x values should include the vertical asymptote at $x = -2$, and be wide enough to show the function approaching its horizontal asymptotes. We use x range $[-15, 15]$. The range of y values should include the asymptotes $y = 1$ and $y = -1$, but also be wide enough to show that function growing toward its vertical asymptote at $x = -2$. We use y range $[-5, 5]$.

We need to sketch a graph that is

- Asymptotic to $y = -1$ for large negative x
- Decreasing on $(-\infty, -2)$
- Approaching $-\infty$ as x approaches -2 from the left
- Not defined at $x = -2$, which is a vertical asymptote
- Approaching $+\infty$ as x approaches -2 from the right
- Decreasing on $(-2, 1)$
- Local minimum at $(1, 0)$
- Increasing on $(1, \infty)$

- Asymptotic to $y = 1$ for large positive x

Incorporating all of these features, we find a graph as below. (Note that this graph includes accurate concavity information, which was not requested in the problem.)



Below are the problems that were graded and the scoring system that was used for each problem.

4. [10 points] The radius of a cylinder is decreasing at a rate of 4 ft/min, while the height is increasing at a rate of 2 ft/min. Find the rate of change in the volume when the radius is two feet and the height is six feet.

2 points – If the student has the formula $V\ pir^2h$
2 points – If the student properly differentiates with respect to t and finds
$\frac{dV}{dt} = \pi r^2 \frac{dh}{dt} + 2\pi r h \frac{dr}{dt}$
2 points – If the student properly labels SOME of the information given in the problem (i.e. $r = 2$, $\frac{dr}{dt} = -4$, etc)
OR
4 points – If the student properly labels ALL of the information given in the problem (i.e. $r = 2$, $\frac{dr}{dt} = -4$, etc)
2 points – If the student gives the correct final answer (units included)

6. [10 points] Consider the function $g(x) = -|x| + 1$ for $-1 \leq x \leq 1$.

1. Draw g on the domain given.
2. A rectangle is to be formed with one side on the x-axis, and the vertices of the opposite side lying on g. Find the maximum area of such a rectangle.

(a)
0 points – If the graph is incorrect
2 points – If the graph is correct
(b)
If the graph in part (a) is incorrect, the student cannot earn any points on part (b), so the following points can only be earned if part (a) is correct.
2 points – If the student writes SOME function representing area that is incorrect (this is the maximum number of points the student can earn on part (b))
OR
4 points – If the student has a correct function representing area $A(x)$ (in terms of one variable)
1 point – If the student has the proper domain for the area function $A(x)$
2 points – If the student differentiates properly to find the correct value for x that maximizes $A(x)$
1 point – If the student comes to the correct conclusion (with proper units)