Math 180 Written Homework

Assignment #7

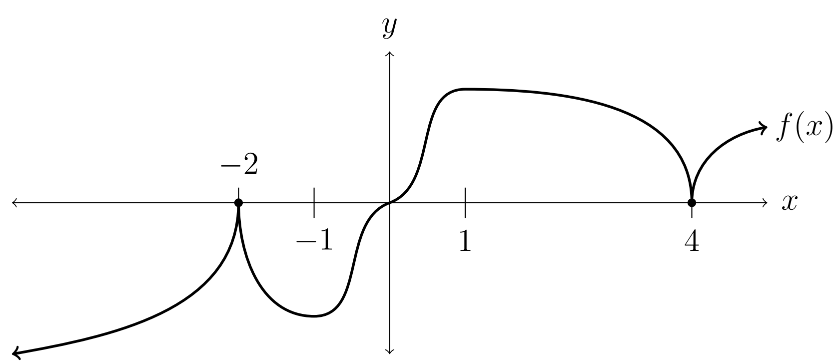
Due **Tuesday, October 21st** at the beginning of your discussion class.

Directions. You are welcome to work on the following problems with other MATH 180 students, but your solutions must be hand-written, by your own hand, representing your understanding of the material. Word-by-word copying from another student or any other source is unacceptable. Any work without the proper justification will receive no credit. The list of problem solutions is to be submitted to your TA at the beginning of the discussion class listed above. No late homework will be accepted.

1. Sketch the graph of a function that is continuous on $(-\infty, \infty)$ and satisfies all the following conditions:

   - $f'(x) > 0$ and $f''(x) > 0$ on $(-\infty, -2)$;
   - $f'(x) > 0$ and $f''(x) < 0$ on $(4, \infty)$;
   - $f'(-2)$ and $f'(4)$ do not exist;
   - $f'(-1) = f'(1) = 0$ and $f''(0) = 0$.

   Solution: Here is one such graph.



2. Find the absolute maximum and minimum values of the function $f(x) = 3|x| - x^3$ on the interval $[-1, 2]$.

   Solution: We seek the critical points of f on the interval $(-1, 2)$.

   - When $x > 0$ we have $f(x) = 3x - x^3$ and $f'(x) = 3 - 3x^2$. The derivative vanishes when $x = 1$. Thus, $x = 1$ is a critical point.
   - When $x < 0$ we have $f(x) = -3x - x^3$ and $f'(x) = -3 - 3x^2$. The derivative will never vanish in this case.

The derivative does not exist at $x = 0$ because it approaches 3 as $x \to 0^+$ and -3 as $x \to 0^-$. Thus, $x = 1$ and $x = 0$ are the critical points of f on $(-1, 2)$.

Next, we evaluate f at the critical points and at the endpoints of the interval.

- $f(1) = 2$
- $f(0) = 0$
- $f(-1) = 4$
- $f(2) = -2$

The absolute maximum of f is 4 and the absolute minimum of f is -2 on the interval $[-1, 2]$.

3. Let $f(x)$ be the function defined on the interval $[0, 2\pi]$, whose derivative is

$$f'(x) = (\sin x + 1)(2\cos x + \sqrt{3}).$$

(a) What are critical points of f?

(b) Determine the intervals on which f is increasing and decreasing.

(c) At what points, if any, does f have local maximum and local minimum values?

SOLUTION: (a) By setting $f'(x) = 0$,

$$\begin{aligned}(\sin x + 1)(2\cos x + \sqrt{3}) = 0 \ &\Rightarrow \ \sin x + 1 = 0 \text{ or } 2\cos x + \sqrt{3} = 0 \\ &\Rightarrow \ \sin x = -1 \text{ or } \cos x = -\frac{\sqrt{3}}{2} \\ &\Rightarrow \ x = \frac{3\pi}{2} \text{ or } x = \frac{5\pi}{6}, \frac{7\pi}{6}\end{aligned}$$

Therefore critical points are $x = \dfrac{5\pi}{6}, \dfrac{7\pi}{6}, \dfrac{3\pi}{2}$.

(b) If $f'(x)$ changes sign, then it does so at critical points and nowhere else; that is, $f'(x)$ has the same sign throughout each of the intervals $\left(0, \dfrac{5\pi}{6}\right)$, $\left(\dfrac{5\pi}{6}, \dfrac{7\pi}{6}\right)$, $\left(\dfrac{7\pi}{6}, \dfrac{3\pi}{2}\right)$, and $\left(\dfrac{3\pi}{2}, 2\pi\right)$. Evaluating $f'(x)$ at selected points of each interval determines the sign of $f'(x)$ on that interval.

- At $x = \dfrac{\pi}{2}$, $f'\left(\dfrac{\pi}{2}\right) = 2\sqrt{3} > 0$, so $f' > 0$ and f is increasing on $\left(0, \dfrac{5\pi}{6}\right)$.

- At $x = \pi$, $f'(\pi) = \sqrt{3} - 2 < 0$, so $f' < 0$ and f is decreasing on $\left(\frac{5\pi}{6}, \frac{7\pi}{6}\right)$.
- At $x = \frac{4\pi}{3}$, $f'\left(\frac{4\pi}{3}\right) = \left(-\frac{\sqrt{3}}{2} + 1\right)(-1 + \sqrt{3}) > 0$, so $f' > 0$ and f is increasing on $\left(\frac{7\pi}{6}, \frac{3\pi}{2}\right)$.
- At $x = \frac{5\pi}{3}$, $f'\left(\frac{5\pi}{3}\right) = \left(-\frac{\sqrt{3}}{2} + 1\right)(1 + \sqrt{3}) > 0$, so $f' > 0$ and f is increasing on $\left(\frac{3\pi}{2}, 2\pi\right)$.

Therefore, f is increasing on $\left(0, \frac{5\pi}{6}\right)$, $\left(\frac{7\pi}{6}, \frac{3\pi}{2}\right)$, $\left(\frac{3\pi}{2}, 2\pi\right)$.
f is decreasing on $\left(\frac{5\pi}{6}, \frac{7\pi}{6}\right)$.
(c) By the First Derivative Test, it follows that

- f has a local maximum value of $f\left(\frac{5\pi}{6}\right)$ at $x = \frac{5\pi}{6}$
- f has a local minimum value of $f\left(\frac{7\pi}{6}\right)$ at $x = \frac{7\pi}{6}$

4. (a) Where is $f(x) = x - \ln x$ increasing?
   (b) Using part (a), prove that $\ln x \leq x$ for $x \geq 1$.

SOLUTION: (a) Since $f'(x) = 1 - \frac{1}{x} > 0$ for $x > 1$, f is increasing for $x > 1$.
(b) $f(1) = 1 - \ln(1) = 1 > 0$. Since f is increasing for $x > 1$ and $f(1) > 0$, $f(x) > 0$ for all $x \geq 1$.

5. Let $f(x) = 3x^5 - 20x^3 + 14$.
   - Find the coordinates of the local extrema and horizontal points of inflection of f.

- Find the intervals of concavity for f.

SOLUTION: (a) $f'(x) = 15x^4 - 60x^2 = 0$. Then $15x^2(x^2-4) = 0$, so $x = 0, \pm 2$. The coordinates of critical points are

$$\begin{aligned} (-2, f(-2)) &= (-2, 78) \\ (0, f(0)) &= (0, 14) \\ (2, f(2)) &= (2, -50) \end{aligned}$$

The first derivative sign diagram is

$$\begin{array}{ccccccc} + & & - & & - & & + \\ \hline & -2 & & 0 & & 2 & \\ & & & f' & & & \end{array}$$

so we know that $(-2, 78)$ is a local maximum, $(0, 14)$ is a horizontal point of inflection, and $(2, -50)$ is a local minimum.

(b) $f''(x) = 60x^3 - 120x = 60x(x^2-2) = 0$, so the possible points of inflection are $x = 0, \pm\sqrt{2}$. The second derivative sign diagram is

$$\begin{array}{ccccccc} - & & + & & - & & + \\ \hline & -\sqrt{2} & & 0 & & \sqrt{2} & \\ & & & f'' & & & \end{array}$$

so f is concave up on $(-\sqrt{2}, 0)$ and $(\sqrt{2}, \infty)$ and f is concave down on $(-\infty, -\sqrt{2})$ and $(0, \sqrt{2})$.

6. Find the absolute extrema of $g(x) = x^4 - 32x^2 - 7$ on $[-5, 6]$ (if it exists).

SOLUTION: $g'(x) = 4x^3 - 64x = 4x(x^2-16) = 0$ so the critical points are $x = 0, \pm 4$, each of which is in the interval given. Then

$$\begin{aligned} g(-5) &= -182 \\ g(6) &= 137 \\ g(0) &= -7 \\ g(-4) &= -263 \\ g(4) &= -263. \end{aligned}$$

The absolute maximum is 137 and the absolute minimum is -263.

7. Find the absolute extrema of $h(x) = \dfrac{8+x}{8-x}$ on $[4,6]$ (if it exists).

SOLUTION: Since h is continuous on $[4,6]$, the Extreme Value Theorem says that the absolute extrema of h will exist on $[4,6]$. Then

$$h'(x) = \frac{(8-x)(1)-(8+x)(-1)}{(8-x)^2} = \frac{16}{(8-x)^2}.$$

There are no critical points of h in $[4,6]$, so we only need to test the endpoints.

$$\begin{aligned} h(4) &= 3 \\ h(6) &= 7 \end{aligned}$$

The absolute maximum is 7 and the absolute minimum is 3.