Math 180 Written Homework Solutions

Assignment #6

Due **Tuesday, October 14th** at the beginning of your discussion class.

Directions. You are welcome to work on the following problems with other MATH 180 students, but your solutions must be hand-written, by your own hand, representing your understanding of the material. Word-by-word copying from another student or any other source is unacceptable. Any work without the proper justification will receive no credit. The list of problem solutions is to be submitted to your TA at the beginning of the discussion class listed above. No late homework will be accepted.

1. Let $y = \cos^{-1} x$.

   (a) Using the fact that $\cos y = x$, find $\dfrac{dy}{dx}$ by implicit differentiation.

   (b) Using the identity $\cos^2 y + \sin^2 y = 1$, write $\sin y$ as a function of x. In other words, solve the equation for $\sin y$.
   (Hint: The range of the function $y = \cos^{-1} x$ is $[0, \pi]$. Therefore $\sin y \geq 0$.)

   (c) Using the previous two steps, find $\dfrac{d}{dx}(\cos^{-1} x)$.

   SOLUTION: (a)

$$\begin{aligned} \cos y &= x, && \text{differentiate with respect to } x \\ -\sin y \cdot \frac{dy}{dx} &= 1, && \text{solve for } dy/dx \\ \frac{dy}{dx} &= -\frac{1}{\sin y}. \end{aligned}$$

   (b)
$$\cos^2 y + \sin^2 y = 1 \quad \Rightarrow \quad \sin y = \pm\sqrt{1 - \cos^2 y} = \pm\sqrt{1 - x^2}.$$
   As the hint says, $\sin y \geq 0$ and therefore we have $\sin y = \sqrt{1 - x^2}$.

   (c)
$$\frac{dy}{dx} = -\frac{1}{\sin y} = -\frac{1}{\sqrt{1 - x^2}}.$$

2. Let f be a function differentiable on $(-\infty, +\infty)$. Find the slope of the curve $y = f^{-1}(x)$ at $(4, 7)$ if the following table of values of f and f' is given

x	0	4	7
$f(x)$	7	2	4
$f'(x)$	5	7	9

SOLUTION: The point $(7, 4)$ is on $y = f(x)$ and $f'(7) = 9$. Therefore $(f^{-1})'(4) = \dfrac{1}{f'(7)} = \dfrac{1}{9}$.

3. Compute the derivative of each function.

   (a) $\ln(2x+1)^3$

   (b) $\log_{10}\left(\dfrac{7}{\sqrt{x+3}}\right)$

   (c) $2^{\ln(x)}$

SOLUTION: (a) Since $\ln(2x+1)^2 = 3\ln(2x+1)$, the derivative is

$$3 \cdot \frac{1}{2x+1} \cdot 2 = \frac{6}{2x+1}$$

(b) Since $\log_{10}\left(\dfrac{7}{\sqrt{x+3}}\right) = \log_{10} 7 - \dfrac{1}{2}\log_{10}(x+3)$, the derivative is

$$-\frac{1}{2} \cdot \frac{1}{\ln(10)} \cdot \frac{1}{x+3}$$

(c) Rewrite $y = 2^{\ln x}$ as $\ln(y) = \ln(x) \cdot \ln(2)$. Then

$$\begin{aligned}
\frac{1}{y} \cdot \frac{dy}{dx} &= \frac{\ln(2)}{x} \\
\frac{dy}{dx} &= \frac{y\ln(2)}{x} \\
\frac{dy}{dx} &= \frac{2^{\ln(x)} \cdot \ln(2)}{x}
\end{aligned}$$

4. Consider the function $f(x) = \cos(3\cos^{-1} x)$.

   (a) What is the domain of the function f?

   (b) Calculate the derivative $f'(x)$.

   (c) Find the equation of the tangent line to the graph of f at $x = \frac{1}{2}$.

   (d) Use the angle addition rules for $\sin(x)$ and $\cos(x)$ to show that $f(x) = 4x^3 - 3x$ on its domain.

   SOLUTION: (a) Since the domain of $\cos^{-1}(x)$ is $[-1, 1]$, the domain of f is also $[-1, 1]$.

   (b) $f'(x) = -\sin(3\cos^{-1} x) \cdot \left(-\dfrac{3}{\sqrt{1-x^2}}\right)$.

   (c) A point on the tangent line is $\left(\dfrac{1}{2}, f\left(\dfrac{1}{2}\right)\right) = \left(\dfrac{1}{2}, -1\right)$. The slope of the tangent line is $f'\left(\dfrac{1}{2}\right) = 0$. The equation is then $y = -1$.

   (d) The triple angle formula for cosine is

$$\cos(3\theta) = 4\cos^3(\theta) - 3\cos(\theta).$$

   Using this formula with $\theta = \cos^{-1} x$, we have

$$f(x) = \cos(3\cos^{-1} x) = 4\cos^3(\cos^{-1} x) - 3\cos(\cos^{-1} x) = 4x^3 - 3x.$$

5. Let $f(x) = 2x^3 - x + 5$. Find $\left(f^{-1}\right)'(6)$.

   SOLUTION: The point $(1, 6)$ is on $f(x)$, and the slope of the tangent line to $f(x)$ at $(1, 6)$ is $f'(1)$. Since $f'(x) = 6x^2 - 1$ and $f'(1) = 5$, then

$$\left(f^{-1}\right)'(6) = \frac{1}{f'(1)} = \frac{1}{5}.$$

6. Use logarithmic differentiation to find $\dfrac{dy}{dx}$ for each of the following functions

(a) $y = (\ln x)^{\ln x}$

(b) $y = \dfrac{(5x^2+2)^6}{(1-4x)^{10}}$

(c) $y = x^{\sin x}$

SOLUTION: (a) Taking the natural log of both sides, we have $\ln y = \ln x \cdot \ln(\ln x)$. Then taking the derivative

$$\begin{aligned} \frac{1}{y} \cdot \frac{dy}{dx} &= \frac{1}{x} \cdot \ln(\ln x) + \ln x \cdot \frac{1}{\ln x} \cdot \frac{1}{x} \\ \frac{dy}{dx} &= y\left(\frac{\ln(\ln x)}{x} + \frac{1}{x}\right) \\ \frac{dy}{dx} &= (\ln x)^{\ln x} \cdot \left(\frac{\ln(\ln x)}{x} + \frac{1}{x}\right) \end{aligned}$$

(b) Taking the natural log of both sides, we have $\ln y = 6\ln(5x^2+2) - 10\ln(1-4x)$. Then taking the derivative

$$\begin{aligned} \frac{1}{y} \cdot \frac{dy}{dx} &= \frac{6}{5x^2+2} \cdot (10x) - \frac{10}{1-4x} \cdot (-4) \\ \frac{dy}{dx} &= y\left(\frac{60x}{5x^2+2} + \frac{40}{1-4x}\right) \\ \frac{dy}{dx} &= \frac{(5x^2+2)^6}{(1-4x)^{10}} \cdot \left(\frac{60x}{5x^2+2} + \frac{40}{1-4x}\right) \end{aligned}$$

(c) Taking the natural log of both sides, we have $\ln y = \sin x \cdot \ln x$. Then taking the derivative

$$\begin{aligned} \frac{1}{y} \cdot \frac{dy}{dx} &= \cos x \cdot \ln x + \sin x \cdot \frac{1}{x} \\ \frac{dy}{dx} &= y \cdot \left(\cos x \cdot \ln x + \frac{\sin x}{x}\right) \\ \frac{dy}{dx} &= x^{\sin x} \cdot \left(\cos x \cdot \ln x + \frac{\sin x}{x}\right) \end{aligned}$$

7. Let $f(x) = \dfrac{1}{4} \cdot 2^x$.

(a) Find $f'(x)$.

(b) Find $f''(x)$.

(c) Find a formula for $f^{(n)}(x)$.

(d) Find a function $g(x)$ such that $g'(x) = f(x)$.

SOLUTION: (a) $f'(x) = \frac{1}{4} \cdot 2^x \ln 2$

(b) $f''(x) = \frac{1}{4} \cdot 2^x (\ln 2)^2$

(c) $f^{(n)}(x) = \frac{1}{4} \cdot 2^x (\ln 2)^n$

(d) $g(x) = \frac{1}{4 \ln 2} \cdot 2^x$

Below are the problems that were graded and the scoring system that was used for each problem.

2. (b) [5 points] Compute the derivative of $\log_{10}\left(\dfrac{7}{\sqrt{x+3}}\right)$

0 points – If the student does not use the Chain Rule

2 points – If the student properly found the derivative of $\log_{10}(x)$ correctly

1 point – If the student properly found the derivative of $\dfrac{7}{\sqrt{x+3}}$

2 points – Only if the student has the correct final answer.

5. [10 points] Let $f(x) = 2x^3 - x + 5$. Find $(f^{-1})'(6)$.

2 points – If the student determines that the point $(1, 6)$ is on f

2 points – If the student finds $f'(x) = 6x^2 - 1$ correctly

1 points – If the student finds $f'(1) = 5$ correctly

5 points – If the student correctly determines that $\left(f^{-1}\right)'(6) = \dfrac{1}{5}$

6. (c) [10 points] Use logarithmic differentiation to find $\dfrac{dy}{dx}$ if $y = x^{\sin x}$.

2 points – If the student rewrites the function as $\ln y = \sin x \cdot \ln x$

3 points – If the student finds the derivative of the left-hand side correctly

3 points – If the student finds the derivative of the right-hand side correctly

2 points – Only if the derivative of both sides is correct and solves for $\dfrac{dy}{dx}$ correctly

OR

2 points – If the student rewrites the function as $y = e^{\sin x \cdot \ln x}$

3 points – If the student uses the Chain Rule and has $e^{\sin x \cdot \ln x}$ in it

3 points – If the student finds the derivative of $\sin x \cdot \ln x$ correctly

2 points – Only if the derivative involves $e^{\sin x \cdot \ln x} = x^{\sin x}$ and multiplies it by the derivative of $\sin x \cdot \ln x$